Unit - 1

Differential Equations of the first order:-

Differential Equations is an equation which invalued derivatives. examples: $y' = \sin x$, $y'' + 3y' + 2y = e^x$, $(y'')^2 + (y')^3 + 3y = x^2$, $y'' + 3(y'')^2 + y' = e^x$

3. $y = xy'' + \partial \sqrt{(1+(y')^2}$, $\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = z$, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y$

Two Type differential Equations

* ordinary differential equations
* partial differential equations

Ordinary differential equations:-

In a differential equation is there is a single indipendant variables and the derivatives are ordinary derivatives. Then it is called an ordinary differential equation. example $y' = \sin x$, $y'' + 3y' + 2y = e^x$, $(y'')^2 + (y')^3 + 3y = x^2$, $y'' + 3(y'')^2 + y' = e^x$

$y = xy'' + \partial \sqrt{(1+(y')^2}$

partial differential Equation

If there are two or more indipendant variables and the derivatives are partial derivatives. Then it is called a partial differential equation. example. $\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = z$, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y$

Define: order

The order of a differential equation is the order of the highest derivatives appearing in it.

example:-

$y' = \sin x$ and $\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = z$ are first order and $y'' + 3y' + 2y = e^{x}$, $(y'')^{2} + (y')^{3} + 3y = x^{2}$, $y = xy'' + \partial\sqrt{1+(y')^{2}}$ are of order two $y''' + 3(y'')^{2} + y' = e^{x}$ is the order three

Define: degree:-

The degree of the differential equation is the degree of the highest order derivatives occuring in it. when the differential ~~coe~~ coefficient are free from radicals and fractions

In the above examples all ~~ex~~ except three and five of degree one. and examples three and five of degree two.

Define:-

A ~~sol~~ solution of a differential equation is a function connecting the variable's which together with the derivatives uptined from its statisfied the given differential equation. Let $f(x, y, y', y'' \dots y^{(n)}) = 0$

Be a differential equation of n th order. A solution of equation ① containing n indipendent orbitary constant is called a jenaral solution

A solution of equation ① uptin from a jenaral solution by giving particular values 2 arpalary constant is called a particular solution of ①

A solution of equation ① which cannot be uptin from any jenaral solution by any choose of the orbitary constant is called a singular solution of ①

Formation of Differential equation:-

Ex:1

Form the differential equation for which $xy = ae^{x} + be^{-x} + x^{2}$ is a solution.

Solu

Let

$$xy = ae^{x} + be^{-x} + x^{2} \longrightarrow ①$$

There are two orbitary constant in ①

Hence, differentiaing ① twice with respect to,

$$x\frac{dy}{dx} + (1)y = ae^{x} - be^{-x} + 2x$$

Again Dwrt x

$$\frac{dy}{dx} + \frac{d^{2}y}{dx^{2}}x + \frac{dy}{dx} = ae^{x} + be^{-x} + 2$$

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = ae^{x} + be^{-x} + 2$$

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = (xy - x^2) + 2$$

~~Which~~ is the required differential equation.

Ex: 2

Show that the differential equation of the family of circles of fixed radius r with centre and y axis is $(x^2 - r^2)(y')^2 + x^2 = 0$

Soln:-

we know the equation of the family of the circles ~~are~~ fixed radius r with centres and y axis is given by ~~...~~ $x^2 + (y-f)^2 = r^2 \rightarrow 1$

Differention with respect to x we get

$$2x + 2(y-f)\frac{dy}{dx} = 0 \rightarrow 2$$

From 2 we get

$$2(y-f)y' = -2x$$

$$y-f = -\frac{x}{y'}$$

subtitiling in 1 weget

$$~~x~~ \quad x^2 + \frac{x^2}{(y')^2} = r^2$$

$$\Rightarrow x^2(y')^2 + x^2 = r^2(y')^2$$

$$\Rightarrow x^2(y')^2 - r^2(y')^2 + x^2 = 0$$

$$\Rightarrow (x^2 - r^2)(y')^2 + x^2 = 0$$

which is the requied differential equation.

Equations of first order and frist degree:-

Type: A

variable seperable:-

The simplest type of a differencial equation of first order and first degree is the case in which the variable and sperable. such an equation is of the form $\frac{dy}{dx} = \frac{F(x)}{g(x)}$ which can be written as $g(y)\,dy = F(x)\,dx$.

Hence $\int g(y)\,dy = \int F(x)\,dx + C$

Ex: 3

Solve $\frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0$

<u>solu</u>

Given, $\frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0$

seperating the variables we get

$$\Rightarrow \frac{dy}{1+y^2} + \frac{dx}{1+x^2} = 0$$

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on integration we get,

$$\int \frac{dy}{1+y^2} + \int \frac{dx}{1+x^2} = 0$$

$$\Rightarrow \tan^{-1} y + \tan^{-1} x = C \quad \Rightarrow \tan^{-1}\left[x(1-y^2) + y(1-x^2)\right]$$

$$\Rightarrow x - xy^2 + y - x^2 y = c$$

Ex : 4

solve $y' = (y/x) + \tan(y/x)$

Soln

given $y' = (y/x) + \tan(y/x)$

put, $z = y/x$

Hence $xz = y$

Differensiate with respect to x, we get

$$x\frac{dz}{dx} + z = \frac{dy}{dx}$$

The given differential equation becomes

$$x\frac{dz}{dx} + z = \tan z + \cancel{z}$$

$$x\frac{dz}{dx} = \tan z$$

$$\frac{1}{\tan z} dz = \frac{dx}{x}$$

$$\cot z \, dz = \frac{dx}{x}$$

Integrating we get

$$\int \cot z\,dz = \int \frac{dx}{x}$$

log

$$\log \sin z = \log x + \log c$$

$$\sin z = cx$$

$$\sin(y/x) = cx$$

Ex: 5

Solve $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

Soln

The given equation can be written as

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$$

$$y - x\frac{dy}{dx} = ay^2 + a\frac{dy}{dx}$$

$$a\frac{dy}{dx} + x\frac{dy}{dx} = y - ay^2$$

$$(a+x)\frac{dy}{dx} = (1-ay)y$$

$$\frac{dx}{a+x} = \frac{dy}{(1-ay)y}$$

$$\frac{1}{y(1-ay)} = \frac{A}{y} + \frac{B}{(1+ay)} \Rightarrow 1 = A(1+ay) + By$$

put $y=0$, $A=1$ and put $y = 1/a$, $B = a$

$$\frac{1}{y(1-ay)} = \frac{1}{y} + \frac{a}{(1+ay)}$$

$$\frac{dx}{a+x} = \frac{dy}{y} + \frac{a\,dy}{1-ay}$$

Integration

$$\log(a+x) = \log y - \log(1-ay) + \log c$$

$$x+a = c\left(\frac{y}{1-ay}\right)$$

Type: B

Equations Homogeneous in x and y

A function $f(x,y)$ is called a homogeneous in x and y of degree n. if $f(\lambda x, \lambda y) = \lambda^n f(x,y)$ for all x, y.

we consider a differentiale equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f, g are homogeneous function of the same degree the equation can be changed into differential equation with sceperable variable by means of substitution $y = vx$

$$\text{Let } h(x,y) = \frac{F(x,y)}{g(x,y)} = \frac{dy}{dx}$$

clearly $h(x,y)$ is homogenous function of degree zero

$$\therefore h(\lambda x, \lambda y) = h(x,y)$$

put, $\lambda = 1/x$

Hence, $h(x,y) = h(1, 1/x) = h(1,v) \to ②$

Now $y = vx \Rightarrow dy/dx = v + \frac{dv}{dx}$

$\Rightarrow h(1,v) = v + x\, dv/dx$. (by 1, 2)

$\therefore \frac{dv}{h(1-v)-v} = \frac{dx}{x}$

which can be integrated, we obtain fined. solution by seperating v by y/x.

Ex: 6

solve $\frac{dy}{dx} = \frac{y^3+3x^2y}{x^3+3xy^2}$

$f(\lambda x, \lambda y) = \frac{\lambda^3 y^3 + \lambda^3 x^2 y}{\lambda^3 x^3 + \lambda^3 xy^2}$

Solu

put $y = vx$

Hence $\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$

$$v + x\left(\frac{dv}{dx}\right) = \frac{v^3x^3 + 3x^2vx}{x^3 + 3xv^2x^2}$$

$$= \frac{x^3(v^3+3v)}{x^3(1+3v^2)}$$

$$v + x\frac{dv}{dx} = \frac{v^3+3v}{1+3v^2}$$

$$x\frac{dv}{dx} = \frac{v^3+3v}{1+3v^2} - v$$

$$x\frac{dv}{dx} = \frac{v^3+3v-(v+3v^3)}{1+3v^2}$$

$$= \frac{v^3+3v-v-3v^3}{1+3v^2}$$

$$= \frac{2v-2v^3}{1+3v^2}$$

$$x\frac{dv}{dx} = \frac{2v(1-v^2)}{1+3v^2}$$

$$\frac{dv(1+3v^2)}{v(1-v^2)} = 2\frac{dx}{x}$$

$$\frac{(1+3v^2)}{v(1-v^2)} = \frac{A}{v} + \frac{B}{(1+v)} + \frac{C}{1-v}$$

$$1+3v^2 = A(1+v)(1-v) + Bv(1-v) + Cv(1+v)$$

put $v=1$

$$1+3v^2 = A(2)(0) + B\,1\,(0) + C(1)(1+1)$$

$$4 = 2C$$

$$C = 2$$

put $v=-1$

$$4 = A(0)(2) + B(-1)(+2) + C(-1)(0)$$

$$B = -2$$

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put $v = 0$

$$1 = A(1)(1) + B(0)(1) + C(0)(1)$$

$$\therefore A = 1$$

$$2\frac{dx}{x} = \frac{1}{v} - \frac{2}{1+v} + \frac{2}{1-v}\, dv$$

$$2\log x = \log v - 2\log(1+v) \cancel{\mp} 2\log(1-v) + \log c$$

$$x^2 = \frac{cv}{(1+v)^2(1-v^2)}$$

$$x^2(1+v)^2(1-v)^2 = cv$$

$$x^2(1-v^2)^2 = cv$$

$$x^2\left(1-\frac{y^2}{x^2}\right)^2 = c\left(\frac{y}{x}\right)$$

$$(x^2-y^2)^2 = cxy.$$

Type : C

None Homogeneous Equation of frist degree in x and y.

we consider an equation of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$$

Case 1

If $ab_1 = ba_1$ the sebsitution $ax+by = v$ radusces the given equation to one in which the variable and seperable.

Case : 2

If $ab_1 \neq ba_1$ then the subsition $x = X+h$ and $y = Y+K$ where h and k are such that $ah + bk + c = 0$ and $a_1h + b_1k + c_1 = 0$ reduces the given equation to a homogenous equation in X and Y which can be solved by using type B. The final solusion is got us replacing X and Y by $x-h$ and $y-k$ respectively.

Ex : 7

Solve : $\dfrac{dy}{dx} = \dfrac{x-y+1}{x+y-3}$

Soln

given $\dfrac{dy}{dx} = \dfrac{x-y+1}{x+y-3}$

put $x = X + h$,

$y = Y + K$,

hence

$$\frac{dy}{dx} = \frac{dY}{dX}$$

Hence the given differatical equation

$$\frac{dY}{dX} = \frac{X - Y + h - K + 1}{X + Y + h + K - 3}$$

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choose h and k such that $h-k+1=0$ and $h+k-3=0$. solving the two equations we get,

$$h-k=-1$$

$$h=-1+k$$

$$k-1+k-3=0$$

$$2k=4$$

$$k=2$$

$$h-2=-1$$

$$h=1$$

$$\frac{dy}{dx}=\frac{x-y}{x+y}$$

Now put $y=vx$

Hence $\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v+x\frac{dv}{dx}=\frac{1-v}{1+v}$$

$$x\frac{dv}{dx}=\frac{1-v}{1+v}-v$$

$$x\frac{dv}{dx}=\frac{1-v-v+v^2}{1+v}$$

$$x\frac{dv}{dx}=\frac{1-2v+v^2}{1+v}$$

$$\frac{1+v}{1-2v-v^2}dv=\frac{dx}{x}$$

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Integration

$$-\frac{1}{2}\log(1-2v-v^2) = \log x + \log c$$

$$(1-2v-v^2)^{-1/2} = c_1 x$$

$$(1-2v-v^2)x^2 = c_2$$

$$\left[1-2\left(\frac{y}{x}\right)-\frac{y^2}{x^2}\right]x^2 = c_2$$

$$x^2-2xy-y^2 = c_2$$

$$\therefore (x-1)^2-2(x-1)(y-2)-(y-2)^2 = c_2$$

$$x^2-2xy-y^2+2x+6y = c$$

Ex: 8

solve $\frac{dy}{dx} = \frac{6x-4y+3}{3x-2y+1}$ (or) $\frac{x-2y+3}{2x+y+3}$

Soln

given $\frac{dy}{dx} = \frac{6x-4y+3}{3x-2y+1}$

Put $3x-2y = v$

Hence $3-2\frac{dy}{dx} = \frac{dv}{dx}$

$$\frac{dv}{dx} = 3-2\left(\frac{2v+3}{v+1}\right)$$

$$\frac{dv}{dx} = \frac{3v+3-4v-6}{v+1}$$

$$\frac{dv}{dx} = -\frac{(v+3)}{v+1}$$

$$dx = -\frac{(v+1)}{v+3}\,dv \qquad \therefore \frac{v+3-2}{v+3} = \frac{v+3}{v+3} - \frac{2}{v+3}$$

$$dx = -\left(1 - \frac{2}{v+3}\right)dv$$

Integration

$$x = -v + 2\log(v+3) + c$$

$$x = (2y-3x) + 2\log(3x-2y+3) + c$$

$\div 2$ both side.

$$2x - y = \log(3x-2y+3) + c$$

Exact differential equation :-

consider the differential equation $M(x,y)dx + N(x,y)dy = 0 \rightarrow (1)$

suppose there exists a function $f(x,y)$ such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$

Then the differential equation takes the form. $\left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy = 0$

i.e $df = 0$

On integration we genarl solusion is given by $f(x,y) = c$. In this case exspresion $Mdx + Ndy$ is said be an exact differential and equation ① is called an exact differential equation

working rule :-

1. verify wether the given equation $Mdx + Ndy = 0$ is exact, ie) verify $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

2. If exact, integrate M with respect to x keeping y as constant.

3 find out those terms in N which are free from x and integrate those terms with respect to y

4 The sum of these two expressions equvated to an orbitary constant is the required genaral solution of the given exact equation

Ex: 9 (+)

solve $\frac{dy}{dx} = \frac{x - 2y + 3}{2x + y + 3}$

Solun

Given $\frac{dy}{dx} = \frac{x - 2y + 3}{2x + y + 3}$

put

$$x = X + h$$

$$y = Y + K$$

$$\left.\begin{aligned} x - 2y &= X + h - 2Y - 2K \\ 2x + y &= 2X + 2h + Y + K \end{aligned}\right\}$$

$$v + x\frac{dv}{dx} = \frac{x - 3y}{2x + y}$$

$$y = vx$$

$$x\frac{dv}{dx} = \frac{1 - 4v - v^2}{2 + v}$$

$$\frac{dx}{x} = \frac{2 + v}{1 - 4v - v^2}\,dv$$

$$\log x = \log(1 - 4v - v^2) + \log c_1$$

$$\left[1 - 4v - v^2\right]\frac{x^3}{x} = c_1$$

$$\left[1 - 4\left(\frac{y}{x}\right) - \left(\frac{y^2}{x^2}\right)\right]\frac{x^3}{x} = c_1$$

$$x^3 - 4x^2y - xy^2 \cdot \frac{1}{x} = c_1$$

$$x = vy$$

$$x = v + y\frac{dv}{dy}$$

$$y\frac{dv}{dy} = \frac{1 - 4v - v^2}{2 + v}$$

$$\frac{dy}{y} = \frac{2 + v}{1 - 4v - v^2}\,dv$$

$$\log y = \log(1 - 4v - v^2) + \log c_2$$

$$\left[1 - 4v - v^2\right]\frac{y^3}{y} = c_2$$

$$\left[1 - 4\frac{x}{y} - \frac{x^2}{y^2}\right]\frac{y^3}{y} = c_2$$

$$y^3 - 4xy^2 - x^2y \cdot \frac{1}{y} = c_2$$

$$\frac{x^3 - 4x^2y - xy^2}{x} = c_1 = \frac{y^3 - 4xy^2 - x^2y}{y}$$

$$x^3 - 4x^2y - xy^2 - y^3 + 4xy^2 + x^2y \cdot \left(\frac{1}{x} \div \frac{1}{y}\right) = c$$

$$\frac{x^3 - 3x^2y - y^3 + 3xy^2}{x + y - 2} = c$$

$$(x - y)^3 = c\,(x + y - 2)$$

$$\therefore \quad \frac{x^3 - y^3 - 3xy\,(x - y)\cdot(x - y)}{xy} = c$$

$$\frac{\left[x^3 - y^3\right] - 3xy\,(x^2 + y^2 - 2xy)}{xy} = c \rightarrow \frac{x^3 - y^3 - 3xy\,(x - y)(x - y)}{3y}$$

$$\left(\div\, x + y - 2\right)\quad \frac{(x - y)^3}{x + y - 2} = c$$

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Theorem:-

The differential equation $M dx + N dy = 0$ is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Proof:-

Suppose the equation is exact.

Then $\exists$ a function $f(x, y) \ni : \frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Conversaly, suppose $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow (1)$

we have to construct a function $f(x, y)$ such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$

Let $f = \int M dx + g(y) \rightarrow (2)$

where g is an arbitrary function of y

By definition of f, we have $\frac{\partial f}{\partial x} = M$

Hence the problem is to determine $g(y)$ in such a way $\frac{\partial f}{\partial y} = N$

i.e) $\frac{\partial}{\partial y}\left[\int M\,dx + g(y)\right] = N$

i.e) $\frac{\partial}{\partial y}\left(\int M\,dx\right) + g'(y) = N$

$\therefore g'(y) = N - \frac{\partial}{\partial y}\left(\int M\,dx\right)$

If we prove that the right hand side of (3) is a function of y only than we can integrate (3) to $g(y)$

Now $\frac{\partial}{\partial x}\left[N - \frac{\partial}{\partial y}\left(\int M\,dx\right)\right] = \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x\,\partial y}\left(\int M\,dx\right)$

$$= \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial y\,\partial x}\left(\int M\,dx\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\int M\,dx\right)\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$= 0$ (by (1))

This shows that the right hand side of (3) is a function of y only

Hence (3) $\Rightarrow g(y) = \int\left[N - \frac{\partial}{\partial y}\int (M\,dx)\right]dy$

substituting this value of $g(y)$ in (x)

we get the required function f. Hence the equation is exact and the general solution is given by $f(x,y)=c$

Ex: 11

verify weather $e^y dx + (xe^y+2y)dy=0$ is exact if so slove?

Solu

exact

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

here

$$M = e^y, \quad N = (xe^y+2y)$$

$$\frac{\partial M}{\partial y} = e^y \quad \frac{\partial N}{\partial x} = e^y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial N}$$

Hence the equation is exact.

The given equation can be grouped as

$$(e^y dx + x e^y dy) + 2y\,dy = 0$$

i.e) $d(xe^y) + d(y^2) = 0$

$$xe^y + y^2 = 0$$

Ex: 12

Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

Solu

here

$$M = x^2 - 4xy - 2y^2$$

$$N = y^2 \ldots$$

$$\frac{\partial M}{\partial y} = -4y - 4x - 4y$$

$$\frac{\partial N}{\partial x} = -4y - 4x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the equation is exact

Now

$$\int M\,dx = \frac{x^3}{3} - 2x^2y - 2y^2x$$

In N, the term free from x is y^2 whose integral with respect to y is $y^3/3$

Hence the complete solution is,

$$\tfrac{1}{3}x^3 - 2x^2y - 2y^2x + \tfrac{1}{3}y^3 = c'$$

$$\text{i.e.) } x^3 + y^3 - 6xy(x+y) = 3c' = c$$

Ex:13

solve $x\,dx + y\,dy - \left(\frac{x\,dy - y\,dx}{x^2+y^2}\right) = 0$

soln

given

$$x\,dx + y\,dy - \left(\frac{x\,dy - y\,dx}{x^2+y^2}\right) = 0$$

$$x\,dx + y\,dy - \frac{x\,dy}{x^2+y^2} + \frac{y\,dx}{x^2+y^2} = 0$$

$$\left(x + \frac{y}{x^2+y^2}\right)dx + \left(y - \frac{x}{x^2+y^2}\right)dy = 0$$

$$M = x + \frac{y}{x^2+y^2}, \quad N = y - \frac{x}{x^2+y^2}$$

$$u/v = \frac{v'u - u'v}{v^2}$$

$$\frac{\partial M}{\partial y} = \frac{(x^2+y^2)'y - y'(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{2y \cdot y - (x^2+y^2)}{x^4+y^4+2x^2y^2}$$

$$= \frac{2y^2 - x^2 - y^2}{x^4+y^4+2x^2y^2} = \frac{y^2-x^2}{x^4+y^4+2x^2y^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \left(\because -\frac{(x^2+y^2)'x - x'(x^2+y^2)}{(x^2+y^2)^2}\right)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Now $\int M\,dx = \frac{1}{2}x^2 + \tan^{-1}(x/y)$

In N the term free then x is y where intergal with respect to y is $\frac{1}{2}y^2$

Hence the complete solution

$$\frac{1}{2}x^2 + \frac{1}{2}y^2 + \tan^{-1}(x/y) = c'$$

$$\text{i.e)}\ x^2 + y^2 + 2\tan^{-1}(x/y) = C$$

Theorem:- If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(x)$ a function of x only then $\mu = e^{\int g(x)dx}$ is an I.F of $M dx + N dy = 0$

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Integrating factors :-

Consider the differential equation $y\,dx - x\,dy = 0$ now $M = y$, $N = -x$ and $\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = -1$ Hence the equation is not exact. However when the equation is multiplied by $\frac{1}{x^2}$ we get $\frac{y\,dx - x\,dy}{x^2} = 0$, i.e $d\left(\frac{y}{x}\right) = 0$. Hence the equation became exact.

Ex : 14

Solve : $(x^2 + y^2 + x)\,dx + xy\,dy = 0$

Soln

Here $M = x^2 + y^2 + x$

$N = xy$

$$\frac{\partial M}{\partial y} = 2y \quad , \quad \frac{\partial N}{\partial y} = y$$

Hence the differential equation is not exact

Now,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2y - y}{xy}$$

$$= \frac{1}{x}$$

which is a function of x only. Hence I.F

$$= e^{\int (1/x)\,dx} = e^{\log x} = x.$$

Multiplying the given differential equation by x we get.

$$x\left((x^2 + y^2 + x)\,dx + xy\,dy\right) = 0$$

And this equation is exact.

$\therefore$ The equation can be written as

$$d\left(\tfrac{1}{4}x^4 + \tfrac{1}{3}\cancel{x}^3 + \tfrac{1}{2}x^2y^2\right) = 0$$

$\therefore$ The solution is $\frac{1}{4}x^4 + \frac{1}{3}\cancel{x}^3 + \frac{1}{2}x^2y^2 = c$

Ex: 15

$$(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$$

Soln

Here $M = 2xy^4e^y + 2xy^3 + y$

$N = x^2y^4e^y - x^2y^2 - 3x$

$$\frac{\partial M}{\partial y} = 8xy^3e^y + 6xy^2 + 1 + 2xy^4e^y$$

$$\frac{\partial N}{\partial \cancel{x}} = \cancel{4x^2y^4e^y - 2x^2y + x^2y^4e^y}$$

$$2xy^4e^y - 2xy^2 - 3$$

If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = h(y)$ is a function of y only then $\mu = e^{-\int h(y)dy}$ is an I.F of $Mdx + Ndy = 0$

$$= e^{-\int h(y)\,dy}$$

$$= \cancel{\frac{1}{2xy^4e^y + 2xy^3 + y}\left[8xy^3e^y + 2xy^4e^y + 6xy^2 + 1 - 4x^2y^3e^y + 2x^2y - x^2y^4e^y\right]}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 8xy^3e^y + 8xy^2 + 4$$

$$= \frac{8xy^3e^y + 8xy^2 + 4}{2xy^4e^y + 2xy^3 + y} = \frac{4(2xy^3e^y + 2xy^2 + 1)}{y(2xy^3e^y + 2xy^2 + 1)} = \frac{4}{y}$$

$$= e^{-\int \frac{8xy^3e^y + 8xy^2 + 4}{2xy^4e^y + 2xy^3 + y} dy} = e^{-\int \frac{4}{y} dy} = e^{-4\log y} = e^{\log y^{-4}}$$

$$= e^{-\left[\frac{2xy^4e^y + 8/3\, xy^3 + 8xy^3e^y + 4y}{6xy^3e^y + 4xy^2 + 2xy^4e^y + 1}\right]} + C$$

$$= e^{\log\left(\frac{1}{y^4}\right)} = \frac{1}{y^4}$$

$$(2xy^4e^y + 2xy^3 + y)\,dx = -(x^2y^4e^y - x^2y^2 - 3x)\,dy$$

$$\frac{x^6(y^5e^y + 5y^4e^y)}{5} - \frac{x^6y^3}{3} = 0$$

$$\frac{3x^6y^5e^y + 15x^6y^4e^y - 5x^6y^3}{15} = 0$$

$\times 15$ $\quad 3x^6y^5e^y + 15x^6y^4e^y - 5x^6y^3 = 0$

Ex: 16

solve $\dfrac{dy}{dx} = \dfrac{y^2}{1-xy}$

Solu

$$y^2dx - dy(1-xy) = 0$$

$M = y^2$

$N = -(1-xy)$

$$\frac{\partial M}{\partial y} = 2y \quad , \quad \frac{\partial N}{\partial x} = y$$

Hence the equation is not exact

Now $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2y-y}{y^2} = 1/y$ a function of y only.

$$I.F = e^{-\int (1/y)\,dy}$$

$$= e^{-\log y} = 1/y$$

multiplying the equation by $1/y$. we get the exact equation

$$y\,dx - \left(\frac{1-xy}{y}\right)dy = 0$$

i.e) $(y\,dx + x\,dy) - (\frac{1}{y})dy = 0$

$d(xy) - d(\log y) = 0$

$xy - \log y = c$

Theorem:-

If $M\,dx + N\,dy = 0$ is a homogeneous equation where M and N homogeneous function of degree n and if $Mdx + Ndy \neq 0$ then $\frac{1}{Mx + Ny}$ is IF of the given equation

---

Ex: 17

Solve $(x^3 - 3xy^2)dx - (y^3 - 3x^2y)dy = 0$

Solu

Here

$M = x^3 - 3xy^2$

$N = -(y^3 - 3x^2y)$

We have

$Mx + Ny = x^4 - 3x^2y^2 - y^4 + 3x^2y^2$

$= x^4 - y^4$

Here

$$\frac{1}{Mx + Ny} = \frac{1}{x^4 - y^4}$$

Multiplying the differential equation by $\frac{1}{x^4 - y^4}$, we get the exact equation, $M_1dx + N_1dy = 0$

$\therefore M_1 = \frac{x^3 - 3xy^2}{x^4 - y^4}$ and $N_1 = \frac{-y^3 - 3x^2y}{x^4 - y^4}$

Integrating M_1 with respect to x

$$M_1 = \int \frac{x^3 - 3xy^2}{x^4 - y^4}\,dx$$

$$= \frac{1}{4}\log(x^4 - y^4) - \frac{3}{4}\log\left(\frac{x^2 - y^2}{x^2 + y^2}\right)$$

Hence the solu is

$$\frac{1}{4}\log(x^4 - y^4) - \frac{3}{4}\log\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \log c_1$$

$$\log(x^4 - y^4) = \log\left[c\left[\frac{x^2 - y^2}{x^2 + y^2}\right]^3\right]$$

$$(x^4 - y^4)(x^2 + y^2)^3 = c\,[x^2 - y^2]^3$$

$$(x^2 - y^2)(x^2 + y^2)\cdot(x^2 + y^2)^3 = c\,(x^2 - y^2)^3$$

$$(x^2 + y^2)^4 = c\,(x^2 - y^2)^2$$

Theorem

If $M\,dx + N\,dy = 0$ is of the form $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$ where $f(xy) \neq g(xy)$ then $\frac{1}{Mx - Ny}$ is an I.F of $M\,dx + N\,dy = 0$

---

problem 18

Solve $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$

Solu:-

Here equation of the form

$y\,f(x,y)\,dx + x\,g(xy)\,dy = 0$

$M = y(xy + 2x^2y^2)$

$N = x(xy - x^2y^2)$

we have

$Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3$

$= 3x^3y^3$

$\frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$

$M_1 = \frac{y(xy + 2x^2y^2)}{3x^3y^3}$

$N_1 = \frac{x(xy - x^2y^2)}{3x^3y^3}$

Intergrating M_1 with respect to x, we get

$$M_1 = -\frac{1}{3xy} + \frac{2}{3}\log x$$

In N, the term free from x is $\left(-\frac{1}{3y}\right)$

Hence an intergration with respect to y, we get

$$-\frac{1}{3}\log y$$

The soln $-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$

$$\log\left(\frac{x^2}{y}\right) - \frac{1}{xy} = c$$

Linear Equation

A differential equation is said to be linear if the diffelen variables and its derivates affer only in the first degree. Hence ~~the~~ linear equation of first order is of the form

$\frac{dy}{dx} + Py = Q$. where P, Q are funtions of x alone

Theorem:-
General solution of the linear equation $\frac{dy}{dx} + Py = Q$ is given by $y e^{\int P dx} = \int e^{\int P dx} Q\, dx + c$

Ex:1

solve $\frac{dy}{dx} - y \cot x = 2x \sin x$

soln

Here

$P = -\cot x$ and $Q = 2x \sin x$

$$\int P dx = -\int \cot x \, dx$$

$$= -\log \sin x$$

$$= +\log \operatorname{cosec} x$$

$$e^{\int P dx} = e^{\log \operatorname{cosec} x}$$

$$= \operatorname{cosec} x$$

The solution is given by

$$y \operatorname{cosec} x = \int \operatorname{cosec} x (2x \sin x) dx + c$$

$$= 2\int x \, dx$$

$$y \operatorname{cosec} x = x^2 + c$$

$$y = x^2 \sin x + c \sin x$$

Ex: 2

solve $(1+y^2)\,dx + (x - \tan^{-1}y)\,dy = 0$

Soln

Given

$$(1+y^2)dx + (x - \tan^{-1}y)dy = 0$$

The equation can be written as

$$\frac{dx}{dy} + \frac{x}{(1+y^2)} = \frac{\tan^{-1}y}{1+y^2}$$

which is the linear first order differential equation

Here

$$P = \frac{1}{1+y^2} \quad \text{and} \quad Q = \frac{\tan^{-1}y}{1+y^2}$$

$$\int P\,dx = -\int \frac{1}{1+y^2}\,dx$$

$$= \tan^{-1}y$$

Hence $e^{\int P\,dy} = e^{\tan^{-1}y}$

The solution is given by

$$x\,e^{\tan^{-1}y} = \int \left(\frac{e^{\tan^{-1}y}}{1+y^2}\right) \tan^{-1}y\,dy$$

subsutution $t = \tan^{-1}y$ in the right hand side

$$x\,e^{\tan^{-1}y} = \int e^t\,t\,dt$$

$$= e^t(t-1) + c$$

$$= e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$$

$$x = (\tan^{-1}y - 1) + c\,e^{-\tan^{-1}y}$$

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ex : 3

solve $\frac{dy}{dx} + y\cot x = 4x \operatorname{cosec} x$, given that $y = 0$ when $x = \frac{1}{2}\pi$

Solu

Here

$$P = \cot x \quad \text{and} \quad Q = 4x \operatorname{cosec} x$$

$$\int P\,dx = \int \cot x\,dx$$

$$= \log \sin x$$

$$= \log \cancel{\operatorname{cosec}} x$$

$$e^{\int P\,dx} = e^{\log \cancel{\sin} x}$$

$$= \cancel{\quad} \sin x$$

The solution is given by

$$y \sin x = \int \sin x\,(4x \operatorname{cosec} x)\,dx + c$$

$$y \sin x = 2x^2 + c$$

when $x = \frac{1}{2}\pi$; $y = 0 \rightarrow c = -\frac{1}{2}\pi^2$

Hence the required solution

$$y \sin x = 2x^2 - \frac{1}{2}\pi^2$$

Benoulles Equation

The equation $\frac{dy}{dx} + Py = y^n Q$

where P and Q are function of x

is called ~~bernoulli~~ Bernoullis

when $n=0$ or $n=1$ it is already linear. for other values of n it can be ~~redu~~ reduced to a linear equation By substitution $z=y^{1-n}$ with this subsitution the given equation becomes $\frac{dz}{dx}+(1-n)Pz=(1-n)Q$ which is linear.

---

problem:-

solve $xy'+y=y^2\log x$.

sol:

Given equation

$$xy'+y=y^2\log x$$

divided by xy^2 throught we get

$$y^{-2}y'+\frac{y^{-1}}{x}=\frac{1}{x}\log x$$

put

$$y^{-1}=z \quad \text{Hence } y^{-2}y'=-z'$$

$$-\frac{dz}{dx}+\frac{z}{x}=\frac{1}{x}\log x$$

$$\frac{dz}{dx}-\frac{z}{x}=-\frac{1}{x}\log x$$

which is a linear in z

Here

$$P=-\frac{1}{x} \text{ and } Q=-\frac{1}{x}\log x$$

$$\therefore \int P\,dx=-\int\left(\frac{1}{x}\right)dx$$

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$$= -\log x$$

$$e^{\int P dx} = e^{-\log x}$$

$$= 1/x$$

The solution is $z\left(1/x\right) = \int 1/x \left(-1/x\right) \log dx + c$

$$= \int -1/x^2 \log x \, dx + c$$

$u\,dv = uv - \int v\,du$

$u = \log x$ | $v = 1/x$

$du = 1/x$ | $dv = -1/x^2 dx$

$$= 1/x \log x - \int \frac{dx}{x^2} + c$$

$$= \left(1/x\right)\left(\log x + 1\right) + c$$

$$1/xy = \left(1/x\right)\left[\log x + 1\right] + c$$

$$1/y = \log x + 1 + cx.$$

Differential equation first order, higher degree

A differential equation of first order and n^{th} degree of the form ~~where~~.

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_{n-1} p + P_n = 0$$

where $p = dy/dx$. These $P_1, P_2 \dots$ and P_n are function of x and y

---

Type A.

equation solvabale for p suppose left hand side of equation ① can be factories in the factor of the first degree. then equation one becomes

$$(P-R_1)(P-P_2)\dots(P-P_n)=0$$

applive a solusion.

$f_i(x,y,c)=0$ corseponding to componant equation $P-R_i=0$ $i=1,2,\dots$

Then general solution of 1 is given by.

$f_1(x,y,c), f_2(x,y,c)\dots f_n(x,y,c)=0$

Type B

equation solvable for y in this case the equation can be put in the form

$$y=f(x,P)\rightarrow ①$$

Differentiated equation ① with respect to x

$$P=\phi\left(x,P,\frac{dP}{dx}\right)\rightarrow ②$$

which is a first order, first degree different-ial equation ~~whi~~ with variable p and x suppose equation ② can be solved to get the realation

$$\psi(x,P,c)=0\rightarrow ③$$

When elemination P from ① and ③ we get required solution.

Type C

Equation solvable for x in this case the equation can be put in the form.

$x = f(y, p) \to ①$

Differentiating with respect to y we get

$$1/p = \varphi\left(y, p, \frac{dp}{dy}\right) \to ②$$

which is a first order, first degree differential equation with variable p and y suppose the equation ② can we sloved to get a relasion

$$\psi(y, p, c) = 0 \to ③$$

The eleminating p from ① and ③ we get the required solution

Type D

CLAIRAUT'S Equation -

An equation of the form $y = px + f(p) \to ①$ is called the clairaut's equation

Diff. w.r. to x, we get

$$p = p + [x + f'(p)]\left(\frac{dp}{dx}\right)$$

$$\text{i.e.} \Rightarrow (x + f'(p))\left(\frac{dp}{dx}\right) = 0$$

$$x + f'(p) = 0 \text{ (or) } \frac{dp}{dx} = 0$$

Now,

$$\frac{dp}{dx} = 0 \Rightarrow p = c \text{ (a constant)}$$

Hence, the general solution of equation ① is

$$y = cx + f(c) \to ②$$

If $x + f'(p) = 0$, we use this and equation ①

explin the solution

This solution is not enclosed in the general solution equation 2

Such a solution is a called a singular solution

problem : 1

whith usuall rotasion solve $p^2 - 9p + 18 = 0$

solu

The given equation is of first order and second degree and it can be solved for p as

$$p^2 - 9p + 18 = 0$$

$$(p - 6)(p - 3) = 0$$

∴ Its component equation are $p = 6$ and $p = 3$

Now, $p = 6 \Rightarrow \frac{dy}{dx} = 6 \Rightarrow y = 6x + c'$

$p = 3 \Rightarrow \frac{dy}{dx} = 3 \Rightarrow y = 3x + c'$

∴ The general solution is $(y - 6x - c)(y - 3x - c) = 0$

problem : 2

solve $4p^2 - 8p + 3 = 0$

solu

The give equation is of first order and second degree and it can be solved for p as

$$4p^2 - 8p + 3 = 0$$

$$(4p - 6)(4p - 2) = 0$$

$(2p-3)(2p-1)=0$

$p = 3/2, \; p = 1/2$

x ts component eq uation are $p = 3/2$ and $1/2$

NOW $\Rightarrow p = 3/2 \Rightarrow \frac{dy}{dx} = 3/2 \Rightarrow y = \frac{3}{2}x + c$

$\Rightarrow p = 1/2 \Rightarrow \frac{dy}{dx} = 1/2 \Rightarrow y = \frac{1}{2}x + c$

The generel equation solution is

$$(y - \tfrac{3}{2}x - c)(y - \tfrac{1}{2}x - c) = 0$$

---

problem: 3

solve $xyp^2 + (3x^2 - 2y^2)p - 6xy = 0$

Solu

The given eq uation is of first order and second degree and it can be solved for p as

$A = xy, \; B = 3x^2 - 2y^2, \; C = -6xy$

Then $p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{(-3x^2 + 2y^2) \pm \sqrt{(3x^2 - 2y)^2 - 24x^2y^2}}{2xy}$$

$$= \frac{-3x^2 + 2y^2 \pm \sqrt{9x^4 + 4y^4 - 12x^2y^2 + 24x^2y^2}}{2xy}$$

$$= \frac{-3x^2 + 2y^2 \pm \sqrt{9x^4 + 4y^4 + 12x^2y^2}}{2xy}$$

$$= \frac{-3x^2+2y^2 \pm \sqrt{(3x^2+2y^2)^2}}{2xy}$$

$$= \frac{-3x^2+2y^2 \pm (3x^2+2y^2)}{2xy}$$

$$= \frac{-3x^2+2y^2+3x^2+2y^2}{2xy} = \frac{4y^2}{2xy} = \frac{2y}{x}$$

$$= \frac{-3x^2+2y^2-3x^2-2y^2}{2xy} = \frac{-6x^2}{2xy} = \frac{-3x}{y}$$

Its component equation are $p = 2y/x$, $p = \frac{-3x}{y}$

Now

$$\Rightarrow p = 2y/x \Rightarrow dy/dx = 2y/x \Rightarrow \frac{dy}{y} = 2\,dx/x$$

$$\log y = 2\log x + \log c_1 \Rightarrow \log y = \log x^2 c_1$$

$$y = c_1 x^2$$

$$\Rightarrow p = -3x/y \Rightarrow dy/dx = -3x/y \Rightarrow y\,dy = -3x\,dx$$

$$\Rightarrow y^2/2 = \frac{-3x^2}{2} + c_2$$

The general solution is

$$(y - c_1x^2)\left(y^2/2 + 3x^2/2 - c_2\right) = 0$$

problem. 4

solve $xp^2 - 2py + x = 0$

solu

solving for p we get $p = \frac{2y \pm \sqrt{4y^2 - 4x^2}}{2x}$

$$(i.e)\ \frac{dy}{dx} = \frac{y \pm \sqrt{y^2 - x^2}}{x} \longrightarrow (1)$$

This is homogenous differential equation in x and y

To solve this put $y = vx$ then (1) becomes

$$v + x\frac{dv}{dx} = v \pm \sqrt{v^2 - 1}$$

$$x\frac{dv}{dx} = \sqrt{v^2 - 1}$$

separting the variable we get

$$\frac{dv}{\sqrt{v^2 - 1}} = \frac{dx}{x}$$

Integration

$$\cosh^{-1} v = \log x + c$$

$$\therefore \cosh^{-1}\left(\frac{y}{x}\right) = \log x + c$$

$$\frac{y}{x} = \cosh(\log x + c)$$

Hence the solution is

$$y = x \cosh(\log x + c)$$

problem: 5

solve $y - 2px = f(xp^2)$

Solu

$$y = 2px + f(xp^2)$$

Diff with respect to x, we get

$$p = 2p + 2x\frac{dp}{dx} + f'(xp^2)\left[p^2 + 2xp\frac{dp}{dx}\right]$$

$$\therefore p + 2x\frac{dp}{dx} + p.f'(xp^2)\left[p + 2x\frac{dp}{dx}\right] = 0$$

$$\therefore \left[p + 2x\frac{dp}{dx}\right] \times \left[1 + pf'(xp^2)\right] = 0$$

$$p + 2x\,dp/dx = 0 \text{ (or) } \left[1 + pf'(xp^2)\right] = 0$$

$$2\frac{dp}{dx} + dx/x = 0 \qquad p^2 x = c$$

$$2\log p + \log x = \log c_1 \qquad p = \sqrt{c/x}$$

substuting the value p in the given equation we get

$$y = 2\sqrt{c\,x} + f(c)$$

---

problem: 6

solve $3x - y + \log p = 0$

Solu

$$y = 3x + \log p$$

Diff with respect to x, we get

$$\frac{dy}{dx} = p = 3 + 1/p\left(dp/dx\right)$$

$$\therefore p^2 - 3p = dp/dx$$

$$p(p-3) = dp/dx$$

$$\frac{dx}{P(P-3)} = \frac{A}{P} + \frac{B}{P-3}$$

$$dx = (P-3)A + PB$$

$P = 3$

$$1 = 0A + 3B$$

$$B = \frac{1}{3}$$

$P = 0$

$$1 = -3A + 0$$

$$A = \frac{1}{-3}$$

$$= -\frac{1}{3P} + \frac{1}{3(P-3)}$$

$$\frac{dx}{P(P-3)} = \frac{1}{3}\left(\frac{1}{P-3} - \frac{1}{P}\right) dP$$

$$3dx = \left(\frac{1}{P-3} - \frac{1}{P}\right) dP$$

$$3x = \log(P-3) - \log P - \log c_1$$

$$3x = \log\left(\frac{P-3}{cP}\right)$$

$$\frac{P-3}{P} = ce^{3x}$$

$$\frac{3}{P} = 1 - ce^{3x}$$

Hence $P = \dfrac{3}{1-ce^{3x}}$

Sub this value p in the given equation we get the solu

$$y = 3x + \log\left(\frac{3}{1-ce^{3x}}\right)$$

Problem 7

Solve $4y = x^2 + p^2$

soln

$$4y = x^2 + p^2$$

x [illegible]

$$4\frac{dy}{dx} = 2x + 2p\frac{dp}{dx}$$

$$\frac{4dy}{4dx} = 2x + 2p\frac{dp}{dx}$$

$$4p = 2x + 2p\frac{dp}{dx}$$

$\div 4$

$$p = \frac{2x}{4} + \frac{2p}{4}\frac{dp}{dx}$$

$$\frac{2x}{4p} + \frac{2}{4}\frac{dp}{dx} = 0$$

$$\frac{1}{4}\left[\frac{2x}{p} + 2\frac{dp}{dx}\right] = 0$$

$$\frac{2x}{p} + 2\frac{dp}{dx} = 0$$

x [illegible]

$$2\frac{x^2}{2p} + 2x\log p + 2p = 0$$

$$\frac{x^2}{p} + 2x\log p + 2p = 0$$

p [illegible] $\quad x^2 + 2px\log p + 2p^2 = 0$

$$2p^2 + 2px\log p + x^2 = 0$$

solving for p we get $p = \dfrac{-2x \pm \sqrt{4x^2 - 4\times 2\times x^2}}{4}$

$$= \frac{-2x \pm \sqrt{4x^2 - 8x^2}}{4}$$

$$= \frac{-2x \pm \sqrt{-4x^2}}{4} = \frac{-2x \pm 2x}{4}$$

$$= \frac{-2x + 2x}{4} = 0$$

$$= \frac{-2x - 2x}{4} = \frac{-4x}{4} = -x$$

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$$2p^2 + 2px \log p + x^2 = 0$$

$p = 0$

$$0 + 0 + x^2 = 0$$

$$x^2 = 0$$

$$p = 0 \Rightarrow \frac{dy}{dx} = 0 \Rightarrow y = c_1$$

$p = -x$

$$p = -x \Rightarrow \frac{dy}{dx} = -x \Rightarrow dy = -x\,dx$$

$$y = -\frac{x^2}{2} + c_2$$

Hence the solustion is

$$(y - c_1) \cdot \left(y + \frac{x^2}{2} - c_2\right) = 0$$

---

problem : 8

solve $y = 2px + y^2 p^3$

Solu xஐப் பொறுத்து வகையிட

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2yp^3\frac{dy}{dx} + 3p^2y^2\frac{dp}{dx}$$

$$p = 2p + 2x\frac{dp}{dx} + 2p^3y\frac{dy}{dx} + 3p^2y^2\frac{dp}{dx}$$

$$2p^2y\frac{dy}{dx} + 3p^2y^2\frac{dp}{dx} + 2 + 2\frac{x}{p}\frac{dp}{dx} = 0$$

$$2p^3y + 3py^2\frac{dp}{dx} + 2 + 2\frac{x}{p}\frac{dp}{dx} = 0$$

$$\frac{2p^4y^2}{8} + \frac{3p^2y^3}{6} + 2p + \frac{2x^2\log p}{2} = 0$$

$$\frac{p^4y^2}{4} + \frac{p^2y^3}{3} + 2p + x^2\log p = 0$$

$$3p^4y^2 + 4p^2y^3 + 24p + 12x^2\log p = 0$$

Given

$$y = 2px + y^2p^3 \Rightarrow x = \frac{y}{2p} - \frac{y^2p^2}{2}$$

D.w.r.t x

$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\left(\frac{dp}{dy}\right) - yp^2 - py^2\left(\frac{dp}{dy}\right)$$

$$\therefore \frac{1}{2p} + p^2y + \left(\left(\frac{y}{2p^2}\right) + py^2\right)\frac{dp}{dy} = 0$$

$$\left(\frac{1+2p^3y}{2p}\right) + y\left(\frac{1+2p^3y}{2p^2}\right)\frac{dp}{dy} = 0$$

$$\left(\frac{1+2p^3y}{2p}\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

$$1 + \frac{y}{p}\frac{dp}{dy} = 0$$

$$\frac{dy}{y} + \frac{dp}{p} = 0$$

$$\log y + \log p = \log c$$

$$c = py$$

$$p = c/y$$

Hence the solution is

$$y = \frac{2cx}{y} + y^2\left(\frac{c^3}{y^3}\right) \Rightarrow y^2 = 2cx + c^3$$

Ex: 9

solve $x = px + p - p^2$

Sol:

D.w.r.t x

$$p = p + xp' + p' - 2pp'$$

$$p'(x+1-2p) = 0$$

$$p' = 0 \text{ (or) } 2p = x+1$$

$$p = c \text{ (or) } p = \tfrac{1}{2}(x+1)$$

when $p = c$ we get

$$y = cx + c - c^2$$

when $p = \frac{1}{2}(x+1)$ we get

$$y = \tfrac{1}{2}(x+1)x + \tfrac{1}{2}(x+1) - \tfrac{1}{4}(x+1)^2$$

$$4y = (x+1)^2$$

Unit - II

Linear equation with constant coefficients

A linear equation of n^{th} order with constant coefficients is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = x \dots (1)$$

where

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2} \dots \dots D^n = \frac{d^n}{dx^n}$$

Definition :-

The general solution of (1) is of the form

$y = Y + u$ where $Y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is the general solution of (2)

Y is called the complementary function (C.F) and u is called a particular integral (P.I)

Methods of finding complementary function :-

consider the differential equation

$y'' + a y' + b y = 0 \dots (1)$

case (i)

Roots of the A.E are real and distinct say m_1 and m_2

$$C.F = y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

case (ii)

Roots of the A.E are imaginary

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

case (iii)

) Roots of the A.E are real and equal

$$y = e^{mx}(c_1 x + c_2)$$

Methods of finding particular integral

consider $(D^n + a_1 D^{n-1} + \dots + a_n) y = X \rightarrow ①$

$$f(D) y = X$$

Type A :-

X is of the form e^{ax}

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \quad f(a) \neq 0$$

suppose $f(a) = 0$

$$\frac{1}{f(D)} e^{ax} = \frac{1}{\phi(a)} \left(\frac{x^r e^{ax}}{r!}\right)$$

Type B :-

X is of the form $\sin ax$ or $\cos ax$.

$$\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax . \text{ if } f(-a^2) \neq 0$$

D^2 replacing by $-a^2$

Type C :-

X is of the form x^m

Expand $[1/f(D)]$ in ascending power of D as for or as D^m and operate on x^m

Type D

$X = e^{ax} V$ where V is any function of X

$$\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$$

Ex 1

solve $(D^2 - 5D + 6)y = 0$

Solu

give $(D^2 - 5D + 6)y = 0$

The auxiliary equation is $m^2 - 5m + 6 = 0$

$(m-3)(m-2) = 0$. Hence $m = 2, 3$

$C.F = C_1 e^{3x} + C_2 e^{2x}$

The general solution is given by $y = C_1 e^{3x} + C_2 e^{2x}$

Ex: 2

solve $(D^2 + D + 1)^2 y = 0$

Solu

The auxiliary equation is $(m^2 + m + 1)^2 = 0$

$(m^2 + m + 1)(m^2 + m + 1) = 0$

$$= \frac{-b^2 \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$ (twice)

$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

$C.F = e^{-\frac{1}{2}x}\left[C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x\right]$ (twice)

$= e^{-\frac{1}{2}x}\left[(C_1 + C_2 x) \cos \frac{\sqrt{3}}{2} x + (C_3 + C_4 x) \sin \frac{\sqrt{3}}{2}\right.$

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Ex: 3

solve $(D^2-4)y = e^{2x} + e^{-4x}$

Solu

The auxiliary equation is $m^2-4=0$

Hence $m = 2, -2$

$$C.F = C_1 e^{2x} + C_2 e^{-2x}, \quad P.I = \frac{1}{F(D)} e^{x}$$

$$P.I = \frac{1}{D^2-4}\left(e^{2x} + e^{-2x}\right)$$

$$= \frac{1}{D^2-4} e^{2x} + \frac{1}{D^2-4} e^{-2x}$$

$$= \frac{e^{2x}}{(D+2)(D-2)} + \frac{e^{-2x}}{(D+2)(D-2)} \quad \Big[f(D) = 0$$

$$= \frac{1}{(2+2)(D-2)} e^{2x} + \frac{e^{-2x}}{(D+2)(-2-2)}$$

$$= \frac{1}{4(D-2)} e^{2x} - \frac{1}{4(D+2)} e^{-2x}$$

$$= \tfrac{1}{4} x e^{2x} - \tfrac{1}{4} x e^{-2x}$$

The general solu

$$y = C.F + P.I$$

$$y = C_1 e^{2x} + C_2 e^{-2x}$$

Ex: 4

Solve $(D^2 + D + 1)y = \sin 2x$.

The auxiliary equation is $(m^2+m+1) = 0$

(m^2+m+1)

$$m = \frac{-1 \pm i\sqrt{3}}{2}$$

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$$C.F = e^{-\frac{1}{2}x}\left(c_1 \cos \sqrt{3}/_2\, x + c_2 \sin \sqrt{3}/_2\, x\right)$$

$$P.I = \frac{1}{D^2+D+1} \sin 2x$$

$$= \frac{1}{-4+D+1} \sin 2x$$

$$= \frac{1}{D-3} \sin 2x$$

$$= \left(\frac{1}{D-3} \times \frac{D+3}{D+3}\right) \sin 2x$$

$$= \left(\frac{D+3}{D^2-9}\right) \sin 2x$$

$$= \frac{D+3}{-4-9} \sin 2x$$

$$= \frac{D+3}{-13} \sin 2x$$

$$= \frac{D \sin 2x + 3 \sin 2x}{-13}$$

$$= \frac{2 \cos 2x + 3 \sin 2x}{-13}$$

$$y = e^{-\frac{1}{2}x}\left(c_1 \cos \sqrt{3}/_2\, x + c_2 \sin \sqrt{3}/_2\, x\right) + \frac{2 \cos 2x + 3 \sin 2x}{-13}$$

Ex:5

$$(D^3 + 3D^2 + 3D + 1)\, y = e^{-x}$$

soln

The a.e is

$$m^3 + 3m^2 + 3m + 1 = 0$$

$$(m+1)^3 = 0$$

$m = -1$ thrice

$$C.F = (C_1 + C_2 x + C_3 x^2) e^{-x}$$

$$P.I = \frac{1}{(D+1)^3} e^{-x}$$

$$P.I = \frac{x^3}{3!} e^{-x}$$

$$y = (C_1 + C_2 x + C_3 x^2) e^{-x} + \frac{x^3}{3!} e^{-x}$$

Ex:6 solve $y'' + 4y' + 13y = 2e^{-x}$ given $y(0) = 0$ and $y'(0) = -1$

Soln The a.e is $m^2 + 4m + 13 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 4(1)(13)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16-52}}{2}$$

$$= \frac{-4 \pm i6}{2}$$

$$m = -2 \pm 3i$$

$$C.F = e^{-2x}(C_1 \cos 3x + C_2 \sin 3x)$$

$$P.I = \frac{1}{D^2 + 4D + 13} 2e^{-x}$$

Substituting $D = -1$ we get,

$$P.I = \frac{1}{1 - 4 + 13} 2e^{-x} = \frac{1}{5} e^{-x}$$

$$y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x) + \frac{1}{5} e^{-x}$$

Ex: 7 Solve $(D^2+9)y = \cos 3x$.

Soln

$$(D^2+9)\,y = \cos 3x$$

$$m^2+9=0 \Rightarrow m = \pm 3i \quad , \alpha = 0 , \beta = 3$$

$$C.F = (c_1 \cos 3x + c_2 \sin 3x)$$

$$P.I = \frac{1}{D^2+9}(\cos 3x) = \frac{1}{(D+3i)(D-3i)}(\cos 3x)$$

$$\therefore \Rightarrow \frac{1}{D^2+a^2}\cos ax = \frac{-x}{2a}\sin ax$$

$$= \frac{-x}{6}\sin 3x$$

$$y = [c_1 \cos 3x + c_2 \sin 3x - \frac{x \sin 3x}{6}]$$

$$= [c_1 \cos 3x + (c_2 - x/6)\sin 3x]$$

$$y = c_1 \cos 3x + (c_2 - x/6)\sin 3x.$$

Ex: 8 Find the particular integral of $(D^2-4D+3)y = e^x \cos 2x$

Soln

The ae is $m^2 - 4m + 3$

$$P.I = \frac{1}{D^2+4D+3}(e^x \cos 2x)$$

$$\Rightarrow \frac{e^x}{(D+1)^2-4(D+1)+3}\cos 2x$$

$$\Rightarrow \frac{e^x}{D^2+1+2D-4D-4+3}$$

$$\Rightarrow \frac{e^x}{D^2-2D} \cos x$$

$$\Rightarrow \frac{e^x}{-1-2D} \cos x$$

$$\Rightarrow -\frac{e^x}{2D+1} \cos x$$

$$\Rightarrow -\frac{e^x (2D-1)}{4D^2-1} \cos x$$

$$\Rightarrow -e^x \frac{(2D-1)}{4(-1)^2-1} \cos x$$

$$= \frac{-e^x (2D\cos x) - \cos x}{-5}$$

$$= \frac{e^x}{5} 2(-\sin x) - \cos x$$

$$= \frac{-2e^x \sin x - \cos x}{5}$$

x
$\cos x$

Ex:9 solve $(D^2+5D+6)y = x^2$

solu The a.e is $m^2+5m+6=0$

$$(m+3)(m+2)=0 \Rightarrow m = -3, -2$$

$$C.F = C_1 e^{-2x} + C_2 e^{-3x}$$

$$P.I = \frac{1}{D^2+5D+6} x^2$$

$$= \frac{1}{6}\left(\frac{1}{1+\frac{5D+D^2}{6}}\right) x^2$$

$$= \frac{1}{6}\left[1+\frac{5D+D^2}{6}\right]^{-1} x^2$$

$= \frac{1}{6}\left[1-\left(\frac{5D+D^2}{6}\right)+\left(\frac{5D+D^2}{6}\right)^2-\cdots\right](x^2)$

$= \frac{1}{6}\left[1-\frac{5D}{6}-\frac{D^2}{6}+\frac{25D^2}{36}+\right](x^2)$ omitting D^3

$= \frac{1}{6}\left[1-\frac{5D}{6}+\frac{19D^2}{36}\right](x^2)$

$= \frac{1}{6}\left[x^2-\frac{5D(x^2)}{6}+\frac{19D^2(x^2)}{36}\right]$

$= \frac{1}{6}\left[x^2-\frac{5x}{3}+\frac{19}{18}\right]$

$y = C.F + P.I$

$y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{1}{6}\left[x^2-\frac{5x}{3}+\frac{19}{18}\right]$

Evoluate the particular Integral of the differential equation

$(D^2+9)y = 4\sin 3x$

Solu

$P.I = \frac{1}{D^2+9}(4\sin 3x)$

$= 4$ imaginary part of $\left(\frac{1}{D^2+9}\right)e^{i3x}$

$= 4$ imaginary part of $\left(\frac{1}{(D+3i)(D-3i)}\right)e^{i3x}$

$= 4$ imaginary part of $\frac{1}{6i(D-3i)}e^{i3x}$

$= 4$ imaginary part of $\frac{x e^{i3x}}{6i}$

$= 4$ imaginary part of $\frac{-ix}{6}(\cos 3x + i\sin 3x)$

Hom

x^n

thi

$= 4\left(\frac{-x\cos 3x}{6}\right)$

Homogeneous linear equations :-

consider a differential equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_{n-1} x \frac{dy}{dx} + P_n y = X \rightarrow ①$$

this equation is called a homogeneous linear equations

put $z = \log x$

$x = e^z$

i) $x \, dy/dx = \theta$

ii) $x^2 \frac{d^2}{dx^2} = (\theta^2 - \theta) = \theta(\theta - 1)$

iii) $x^3 \, d^3/dx^3 = \theta(\theta-1)(\theta-2)$

iv) $x^n \, d^n/dx^n = \theta(\theta-1)(\theta-2) \dots (\theta - n + 1)$

$P.I = \frac{1}{f(\theta)} X$

Let $\frac{1}{\theta - \alpha} X = y$

Hence $x \, dy/dx - \alpha y = X$.

$$y = x^{\alpha} \int x^{-\alpha-1} X \, dx$$

Ex: 1

Slove $x^2\frac{d^2y}{dx^2} - 3x\,dy/dx - 5y = \sin(\log x)$

<u>solu</u>

given

$$x^2 d^2y/dx^2 - 3x\,dy/dx - 5y = \sin(\log x)$$

put

$$z = \log x \quad , \theta = d/dz$$

$\therefore$ The given equation redues to

$$[\theta(\theta-1)y - 3\theta y - 5y] = \sin z$$

$$y[\theta^2 - \theta - 3\theta - 5] = \sin z$$

$$(\theta^2 - 4\theta - 5)y = \sin z$$

The a.e is $m^2 - 4m - 5 = 0$

$$(m-5)(m+1) = 0$$

$$m = 5, -1$$

g.~~s~~e

$$C.F = c_1 e^{5x} + c_2 e^{-z}$$

$$= c_1 e^{5\log x} + c_2 e^{-\log x} = c_1 x^5 + c_2 x^{-1}$$

$$P.I = \left(\frac{1}{\theta^2 - 4\theta - 5}\right)\sin z$$

$$= \left(\frac{1}{-1-4\theta-5}\right)\sin z$$

$$= \left(\frac{1}{-2(2\theta+3)}\right) \sin z$$

$$= \left(\frac{2\theta-3}{-2(4\theta^2-9)}\right) \sin z$$

$$= \frac{2\theta-3}{26} \sin z$$

$$= \frac{2\theta(\sin z) - 3\sin z}{26}$$

$$= \frac{2\cos z - 3\sin z}{26}$$

$$= \frac{1}{13}\cos\log x - \frac{3}{26}\sin\log x$$

$y = C.F + P.I$

$= c_1 x^5 + c_2 x^{-1} + \frac{1}{13}\cos\log x - \frac{3}{26}\sin\log x$

Ex:2

Solve $x^2y'' - xy' + 4y = \cos(\log x) + x\sin(\log x)$

Soln

put

$z = \log x \quad , \theta = d/dz$

The given equation reduces to

$$[\theta(\theta-1) - \theta + 4]y = \cos z + e^z \sin z$$

$$[\theta^2 - \theta - \theta + 4]\, y = 0$$

$$[\theta^2 - 2\theta + 4]\, y = 0$$

The a.e is $m^2 - 2m + 4 = 0$

$$= \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2}$$

$m = 1 \pm i\sqrt{3}$

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$$C.F = e^{z}(c_1 \cos\sqrt{3}z + c_2 \sin\sqrt{3}z)$$

$$= x\,(c_1 \cos(\sqrt{3}\log x) + c_2 \sin(\sqrt{3}\log x))$$

$$P.I = \frac{1}{\theta^2 - 2\theta + 4}(\cos z + e^z \sin z)$$

$$= \frac{1}{\theta^2 - 2\theta + 4}(\cos z) + \frac{1}{\theta^2 - 2\theta + 4}(e^z \sin z)$$

$$= \frac{1}{3 - 2\theta}\cos z + e^z\left(\frac{1}{(\theta+1)^2 - 2(\theta+1) + 4}\right)\sin z$$

$$= \frac{1}{3 - 2\theta}\left(\frac{3 + 2\theta}{9 - 4\theta^2}\right)\cos z + e^z\left(\frac{1}{\theta^2 + 3}\right)\sin z$$

$$= \frac{3 + 2\theta}{13}\cos z + e^z\frac{\sin z}{2}$$

$$= \frac{3\cos z + 2\theta(\cos z)}{13} + \frac{e^z \sin z}{2}$$

$$= \frac{3}{13}\cos z - \frac{2}{13}\sin z + \frac{e^z \sin z}{2}$$

$$= \frac{3}{13}\cos(\log x) - \frac{2}{13}\sin(\log x) + \frac{x \sin(\log x)}{2}$$

$$y = x\,(c_1 \cos(\sqrt{3}\log x) + c_2 \sin(\sqrt{3}\log x)) + \frac{3}{13}\cos(\log x)$$
$$- \frac{2}{13}\sin(\log x) + \frac{x \sin(\log x)}{2}$$

Ex:3

Solve $x^2 y'' + 4xy' + 2y = e^x$

put

$\theta = d/dz \quad , x = e^z$

The given equation reduces to

$$[\theta(\theta-1)y + 4\theta y + 2y] = e^x = x$$

$$[\theta^2 - \theta + 4\theta + 2]y = x$$

$$[\theta^2 + 3\theta + 2]\, y = x$$

The a.e is $m^2 + 3M + 2$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm \sqrt{1}}{2}$$

$$= \frac{-3 \pm 1}{2} \Rightarrow \frac{-3-1}{2} = -4/2 = -2 \Rightarrow \frac{-3+1}{2} = -2/2 = -1$$

$m = -1, -2$ (or) $1, 2$

$$C.F = C_1 e^x + C_2 e^{2x}$$

$$= C_1 e^{e^x} + C_2 e^{e^{2x}} = C_1 e^x + C_2 e^{2x}$$

P.I

$$= \frac{1}{D^2 + 3D + 2} e^x = \frac{1}{(D+1)(D+2)} e^x$$

$$= \frac{1}{2 \cdot 3} e^x = e^x/6$$

$$y = C.F + P.I$$

$$= C_1 e^x + C_2 e^{2x} + e^x/6$$

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Ex: 4

solve $x^2y'' + 4xy' - 2y = e^x$

Soln

Put

$x = e^z$, $\theta = d/dz$

$$[\theta(\theta-1)y + 4\theta y - 2y] = e^x$$

$$[\theta^2 - \theta + 4\theta - 2]y = e^x$$

$$[\theta^2 + 3\theta - 2]y = e^x$$

$$m^2 + 3m - 2 = 0$$

$a = 1$, $b = 3$, $c = -2$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9+8}}{2}$$

$$= \frac{-3 \pm \sqrt{17}}{2}$$

$$C.F = c_1 e^{\left(\frac{-3+\sqrt{17}}{2}\right)x} + c_2 e^{\left(\frac{-3-\sqrt{17}}{2}\right)x}$$

$$P.I = \frac{1}{D^2 + 3D - 2} e^x$$

$$= \frac{e^x}{3D-1}$$

$$= \frac{3De^x + e^x}{(3D-1)(3D+1)}$$

$$= \frac{3D+1}{9D^2 - 1} e^x = \frac{3e^x + e^x}{8}$$

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$$= c_1 e^{\left(\frac{-3+\sqrt{17}}{2}\right)x} + c_2 e^{\left(\frac{-3-\sqrt{17}}{2}\right)x} + \frac{3}{8}e^x + \frac{1}{8}e^x.$$

$$= c_1 e^{\left(\frac{-3+\sqrt{17}}{2}\right)x} + c_2 e^{\left(\frac{-3-\sqrt{17}}{2}\right)x} + \frac{3+1}{8} e^x$$

$$= c_1 e^{-\left(\frac{3-\sqrt{17}}{2}\right)x} + c_2 e^{-\left(\frac{3+\sqrt{17}}{2}\right)x} + \frac{1}{2} e^x$$

---

example : 5

solve $x^2y'' + 4xy' - 2y = e^x$

soln:-

Given $x^2y'' + 4xy' - 2y = e^x$

Put $z = \log x$ and $\theta = d/dz$

$$(\theta^2 + 3\theta + 2)y = e^x$$

The a.e is

$$m^2 + 3m + 2 = 0$$
$$(m+1)(m+2) = 0$$
$$m = -1, -2$$

Hence.

$$C.F = c_1 e^{-z} + c_2 e^{-2z}$$
$$= c_1 e^{-\log x} + c_2 e^{-2\log x}$$
$$C.F = c_1 x^{-1} + c_1 x^{-2}$$

$$P.I = \frac{1}{(\theta+1)(\theta+2)} e^x = \left(\frac{1}{\theta+1} - \frac{1}{\theta+2}\right) e^x$$
$$= x^{-1}\int e^x dx - x^2 \int x e^x dx$$
$$= x^{-1} e^x - x^{-2}(xe^x - e^x$$
$$= x^{-2} e^x$$
$$y = c_1 x^{-1} + c_2 x^{-2} + x^{-2} e^x$$

Ex: 5 solve $x^2y'' + 3xy' + y = \frac{1}{1-x^2}$ ..

Soln

Given

$$x^2y'' + 3xy' + y = \frac{1}{(1-x)^2}$$

put $z = \log x$ and $\theta = d/dz$

The given equation reduces to

$$[\theta(\theta-1) + 3\theta + 1]\, y = 1/(1+x)^2$$

i.e) $(\theta^2 + 2\theta + 1)\, y = 1/(1+x)^2$

The a.e is

$$m^2 + 2m + 1 = 0$$

$$m = -1, -1$$

Hence

$$C.F = C_1 e^{-z} + C_2 e^{-z}.z$$

$$= (C_1 + C_2 z)\, e^{-z}$$

$$C.F = x^{-1}(C_1 + C_2 \log x)$$

$$P.I = \frac{1}{(\theta+1)^2} \cdot \frac{1}{(1+x)^2}$$

$$= \frac{1}{\theta+1}\, x^{-1} \int \frac{dx}{(1-x)^2}$$

$$= \frac{1}{\theta+1} \left[\frac{1}{x(1-x)}\right]$$

$$= x^{-1} \int \frac{dx}{x(1-x)} = 1/x \int \left(1/x + \frac{1}{1-x}\right) dx$$

$$= 1/x \log \left(x/1-x\right)$$

The solution is

$$y = x^{-1}(C_1 + C_2 \log x) + 1/x \log (x/1-x)$$

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Ex: 6

Solve $(2x+1)^2 y'' - 2(x+1)y' - 12y = 6x$

solu

Put $2x+1 = z$

Hence $\frac{dz}{dx} = 2$ and $x = \frac{z-1}{2}$

Now $y' = \frac{dy}{dz}\frac{dz}{dx}$

$= 2\, dy/dz$

$y'' = 2\frac{d^2y}{dz^2}\frac{dz}{dx}$

$= 4\frac{d^2y}{dz^2}$

Hence the given equation reduces to a linear homogeneous equation

$$4z^2\frac{d^2y}{dz^2} - 4z\frac{dy}{dz} - 12y = 6\left(\frac{z-1}{2}\right) \to ①$$

Now put $u = \log z$ and $\theta = d/du$ and hence the equation ① reduces

$$[4\theta(\theta-1) - 4\theta - 12]y = 3(e^u - 1)$$

$$\Rightarrow (\theta^2 - 2\theta - 3)y = 3/4\,(e^u - 1)$$

A.E $= m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

Here $m = 3, -1$

$$C.F = c_1 e^{3u} + c_2 e^{-u}$$
$$= c_1 e^{3\log z} + c_2 e^{-\log z}$$
$$= c_1 z^3 + c_2 z^{-1}$$
$$= c_1 (2x+1)^3 + c_2 (2x+1)^{-1}$$

$$P.I = \frac{3}{4}\left(\frac{1}{\theta^2 - 2\theta - 3}\right)(e^u - 1)$$
$$= 3/4 \left(\frac{e^u}{-4} + 1/3\right)$$
$$= 3/4 \left(-z/4 + 1/3\right)$$
$$= 3/4 \left(\frac{-(2x+1)}{4} + 1/3\right)$$
$$= 3/8\, x + 1/16$$
$$y = C.F + P.I$$

Linear equations with variable coefficients:

Type: A

consider the differential equation

$$y'' + Py' + Qy = X \rightarrow (1)$$

where P, Q, X are function of x

$$\frac{dq}{dx} + q\left[P + 2/u\left(\frac{du}{dx}\right)\right] = X/u$$

Here $q = dv/dx$

$$\therefore \frac{d^2v}{dx^2} + \frac{dv}{dx}\left(P + \frac{2}{u}\frac{du}{dx}\right) = \frac{X}{u} \rightarrow (2)$$

$$I.F = e^{\int\left[P + \frac{2}{u}\cdot\left(\frac{du}{dx}\right)\right]dx}$$

$$= e^{\int P\,dx + 2\log u}$$

$$= u^2 e^{\int P\,dx}$$

$$V = C_2 + C_1 \int \frac{e^{-\int P\,dx}}{u^2}\,dx + \int\left[\frac{e^{-\int P\,dx}}{u^2}\int u X e^{\int P\,dx}\right]du \rightarrow (3)$$

$$y = uv$$

---

example : 1

solve : $xy'' - (2x+1)y' + (x+1)y = x^2e^x$.

solu

Given equation given be written as

$$y'' - (2 + 1/x)y' + (1 + 1/x)y = xe^x \rightarrow (1)$$

Here $P = -2 - 1/x$

$Q = 1 + 1/x$

$X = xe^x$

Since

$1 + P + Q = 0$, $u = e^x$ in an integral of the

C.F of (1)

Let the general solu of (1) be $y = uv$

Then (1) reduces to $\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\left(\frac{du}{dx}\right)\right]\frac{dv}{dx} = \frac{X}{u}$

$$\frac{d^2v}{dx^2} - \frac{1}{x}\left(\frac{dv}{dx}\right) = x$$

$\therefore \frac{dq}{dx} - \left(\frac{1}{x}\right)q = x$ where $q = \frac{dv}{dx}$

This is linear equation in q and x

Its I.F $= e^{-\int (1/x)dx} = e^{-\log x} = x^{-1}$

Hence $qx^{-1} = \int x\, x^{-1} dx + C_1 = x + C_1$

$$\frac{dv}{dx} = C_1 x + x^2$$

$$V = \frac{1}{2} C_1 x^2 + \frac{1}{3} x^3 + C_2$$

Hence the general solution is $y = uv$

$$= e^x\left(\frac{1}{2}C_1 x^2 + \frac{1}{3}x^3 + C_2\right)$$

example: 2

Give that $y = x$ is a particular solution of the differential equation $x^2 y'' - 2x(1+x)y' + 2(1+x)y = $ find the general solution.

soln

$$y'' - 2\left(1 + \frac{1}{x}\right)y' + 2\left(\frac{1}{x^2} + \frac{1}{x}\right)y = x$$

Here

$$P = -2\left(1 + \frac{1}{x}\right)$$

$$Q = 2\left(\frac{1}{x^2} + \frac{1}{x}\right)$$

$$X = x$$

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Let the general solution of (1) be $y = uv$ where $y = x$

Hence we get $\quad 1 - P + Q = 0$, $u = e^x$ is an

$$\frac{d^2v}{dx^2} - x\frac{dv}{dx} = 1$$

$$\frac{dq}{dx} - 2q = 1 \quad \left[\text{by puting } q = \frac{dv}{dx}\right)$$

This is a linear equation and its

$$I.F = e^{-\int 2dx} = e^{-2x}$$

$$q e^{-2x} = \int e^{-2x} dx + C_1$$

$$= -\tfrac{1}{2} e^{-2x} + C_1$$

$$q = dv/dx = -\tfrac{1}{2} + C_1 e^{2x}$$

$$v = -\tfrac{1}{2}x + \tfrac{1}{2} C_1 e^{-2x} + C_2$$

The general solu of (1) is $y = uv$

$$= x\left(-\tfrac{1}{2}x + \tfrac{1}{2} C_1 e^{-2x} + C_2\right)$$

Ex: 3

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Simultaious equations :-

Taking z as the indepentent variable and x, y as the pare of depentent variable's, a pare of simultaious differential equation of the first order first degree may be written as

$$\left.\begin{aligned} P_1 \frac{dx}{dz} + Q_1 \frac{dy}{dz} + R_1 = 0 \\ P_2 \frac{dx}{dz} + Q_2 \frac{dy}{dz} + R_2 = 0 \end{aligned}\right\} \rightarrow (1)$$

$$\left.\begin{aligned} P_1\, dx + Q_1\, dy + R_1 dz = 0 \\ P_2\, dx + Q_2\, dy + R_2\, dz = 0 \end{aligned}\right\} \rightarrow (2)$$

The reatiesas is,

$$\frac{dx}{Q_1 R_2 - Q_2 R_1} = \frac{dy}{R_1 P_2 - P_2 P_1} = \frac{dz}{P_1 Q_2 - P_2 Q_1}$$

(or)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \rightarrow (3)$$

---

example:;

Solve the equation $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$

Soln:

Given $\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$

Taking first two equation we get.

$$\frac{dx}{\cancel{y}z} = \frac{dy}{\cancel{x}z}$$

$$\frac{dx}{y} = \frac{dy}{x}$$

$$x\,dx = y\,dy.$$

Integrating

$$\int x\,dx - \int y\,dy = 0$$

$$x^2 - y^2 = c_1$$

Taking the first and last terms of the equation we get.

$$\frac{dx}{\cancel{y}z} = \frac{dz}{\cancel{x}y}$$

$$\frac{dx}{z} = \frac{dz}{x}$$

$$x\,dx - z\,dz = 0$$

Integrating

$$x^2 - z^2 = c_2$$

The general solustion is $\phi(x^2-y^2, x^2-z^2) = 0$

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Ex: 2 Solve the equation $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{x(yz-2x)}$

solu

Taking the first two equation

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\log x = \log y + \log c$$

$$\therefore x = c_1 y$$

Substitusion $x = c_1 y$ in the second and third equation

$$\frac{dy}{y^2} = \frac{dz}{c_1 y(yz - 2c_1 y)}$$

$$\frac{dy}{y^2} = \frac{dz}{c_1 y^2(z - 2c_1)}$$

$$c_1 dy = \frac{dz}{z - 2c_1}$$

$$c_1 y = \log(z - 2c_1) + \log c_2$$

$$e^{c_1 y} = c_2(z - 2c_1)$$

$$e^x = c_2\left(z - 2x/y\right)$$

$$\frac{ye^x}{yz - 2x} = c_2$$

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simultaious linear equations with constant coefficients:-

A pair simultaious linear differential equation can be written as

$$f_1(D)x + \phi_1(D)y = T_1$$

$$f_2(D)x + \phi_2(D)y = T_2$$

general solustion is

$$x = F(t, c_1, c_2 \ldots)$$

---

ex :1

solve the equation $2\frac{dx}{dt} + x + \frac{dy}{dt} = \cos t$

$\frac{dx}{dt} + 2\frac{dy}{dt} + y = 0$

Solu

Given equation is

$$2\frac{dx}{dt} + x + \frac{dy}{dt} = \cos t, \quad \frac{dx}{dt} + 2\frac{dy}{dt} + y = 0$$

Here $d/dt = D$, Then

The equations are

$$2Dx + x + Dy = \cos t$$

$$(2D+1)x + Dy = \cos t \rightarrow ①$$

$$Dx + (2D+1)y = 0 \rightarrow ②.$$

we elemenating x by oparating with D on 1 and with $(2D+1)$ on 2

$D(2D+1)x + D^2y = -\sin t$

$D(2D+1)x + (2D^2+1)^2 y = 0$

substracting the first equation from the second equation we get

$(3D^2+4D+1)y = \sin t \rightarrow ③$

$\therefore \cancel{(2D^2+1)^2 y - D^2 y = \sin t}$ $D^2y - (2D+1)^2 y = \sin t$

$\cancel{4D^4 +}$ $\cancel{D^2 y - (4D^2) y}$

$(D^2 - (2D+1)^2)y = -\sin t$

$(D^2 - 4D^2 - 1 - 4D)y = -\sin t$

$(-3D^2 - 1 - 4D)y = -\sin t$

$(3D^2 + 4D + 1)y = \sin t \rightarrow ③$

Hence

$3m^2 + 4m + 1 = 0$

$= \frac{-b \pm \sqrt{b^2-4ac}}{2ac} = \frac{-4 \pm \sqrt{16-12}}{6} = \frac{-4 \pm \sqrt{4}}{6}$

$= \frac{-4 \pm 2}{6}$

$m = -1, -\frac{1}{3}$

$C.F = Ae^{m_1 x} + Be^{m_2 x}$

$= Ae^{-t} + Be^{-t/3}$

$P.I = \frac{1}{f(D)} G(x)$

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$$= \frac{1}{3D^2+4D+1} \sin t$$

$$= \frac{1}{3(-1)+4D+1} \sin t$$

$$= \frac{1}{4D-2} \sin t$$

$$= \frac{1}{2(2D-1)} \sin t$$

$$= \frac{1}{2} \frac{(2D+1)\sin t}{(2D)^2-1}$$

$$= \frac{1}{2} \frac{2D(\sin t)+\sin t}{4D^2-1}$$

$$= \frac{1}{2} \frac{2\cos t+\sin t}{-5}$$

$$= -\frac{1}{10}(2\cos t+\sin t)$$

$$y = C.F + P.I$$

$$= Ae^{-t} + Be^{-t/3} - \frac{1}{10}(2\cos t+\sin t)$$

Total Differential equations

In a total Differential equations we have the differential coefficient of severale endepented variable with regards to a single indepented variable.

such an equation in three variables referenced by

$Pdx + Qdy + Rdz = 0$

where P, Q, R are functions of x, y, z

$$\therefore P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \quad (2)$$

If relation I exists among P, Q, R a simillar relation holds good between the coefficents of

$$\mu P dx + \mu Q dy + \mu R dz = 0$$

where μ is any function of x, y, z.

Hence, $\mu Pdx + \mu Pdy$ (where z is a parpuler)

Ex :)

solve $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$

solu

Given

$$(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0 \rightarrow (1)$$

Here the contision of integrating is satisified

Negleting $(y^2 - xz)dz$ we get

$$(y^2 + yz)dx + (xz + z^2)dy = 0$$

$$\frac{dx}{(x+z)} + \frac{z\,dy}{y(y+z)} = 0$$

Integrating on the assumpeter there z is constant

$$\int \frac{dx}{(x+z)} + \int \frac{z\,dy}{y(y+z)} = 0$$

$$\log(x+z) + \log\frac{y}{y+z} = \log c$$

$$\log(x+z) + \log\frac{y}{y+z} = \log c$$

$$\frac{(x+z)y}{y+z} = c = f(z) \rightarrow ②$$

Diff (2) totally with respect to x, y, z

$$\frac{(y+z)\left[(dx+dz)y + (x+z)dy\right] - y(x+z)(dy+dz)}{(y+z)^2} = f(z)\,dz$$

$$(y^2+yz)dx + (xz+z^2)dy + dz(y^2 - xy - f(z)(y+z)^2)$$

Comparing this with ① we have $f(z)(y+z)^2 dx = 0$

As $(y+z^2)dz \neq 0$ $f(z) = 0$ $f(z) = \text{constant}$

Hence the integral of ① is $y(x+z) = c(y+z)$