Vertex Colourings

Defn: A k-vertex colouring of G is assignment of k colours $1, 2, \ldots, k$ to the vertices of G. The colouring is said to be proper if no two distinct adjacent vertices have the same colour. Thus a proper k-vertex colouring of a loopless graph G is a partition of V into k independent (possibly empty) sets $(V_1, V_2, \ldots, V_k)$. G is k-vertex colourable if G has a proper k-vertex colouring.

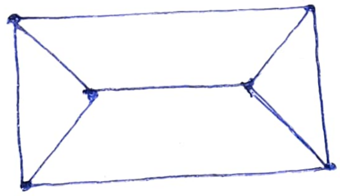
Note: For convenience, we shall abbreviate "proper vertex colouring" as simply a "colouring" and "proper k-vertex colouring" as simply a "k-colouring" and "k-vertex colourable" as "k-colourable."

Remark: i) G is k-colourable if and only if its underlying graph is k-colourable. Hence, in discussing colourings, we shall restrict ourselves to simple graphs.

(ii) G is 1-colourable if and only if it is empty.

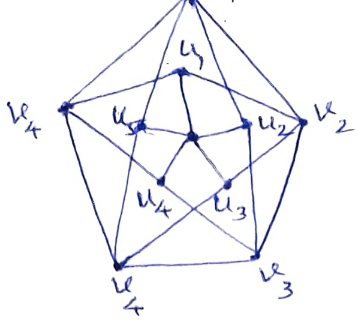
iii) G is 2-colourable if and only if it is bipartite.

Defn The ~~chr~~ chromatic number $\chi(G)$ of a graph G is the minimum k for which G is k-colourable. If $\chi(G) = k$, G is said to be k-chromatic. A 3-chromatic graph is shown in the figure



3-chromatic graph.

Defn: A graph G is critical if $\chi(H) < \chi(G)$ for every proper subgraph H of G. A k-critical graph is one that is k-chromatic and critical. A 4-critical graph is shown in the figure



Grotzsch graph, a 4-Critical graph

Remark i) Every k-chromatic graph has a k-Critical subgraph.

(ii) Every Critical graph is Connected.

Theorem ① If G is k-Critical, then $\delta \geq k-1$

Proof: We prove theorem by the method of Contradiction. If possible, Let G be a k-Critical graph with $\delta < k-1$. Let v be a vertex of degree δ in G. Since G is k-Critical, $G-v$ is $(k-1)$-Colourable. Let $(V_1, V_2, \ldots, V_{k-1})$ be a $(k-1)$-Colouring of $G-v$. From the definition, v is adjacent in G to $\delta < k-1$ vertices and therefore, v must be adjacent in G to every vertex of ~~some~~ some V_j. But, then, $(V_1, V_2, \ldots, V_j \cup \{v\}, \ldots V_{k-1})$ is a $(k-1)$-Colouring of G, a contradiction of G is k-Critical. Thus $\delta \geq k-1$.

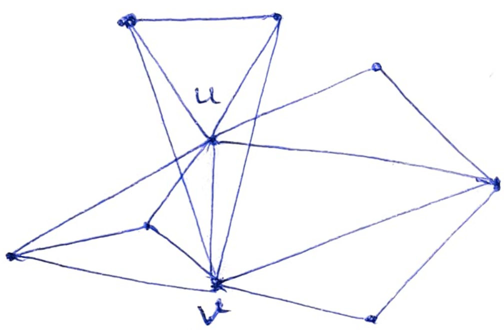
Corollary 1: Every k-Chromatic graph has atleast k vertices of degrees atleast $k-1$.

Proof: Let G be a k-Chromatic graph and let H be a k-critical subgraph of G. From the theorem①, each vertex of H has degree atleast $k-1$ in H and hence, also in G. Being a k-Chromatic graph. G has at least k vertices of degree atleast $k-1$

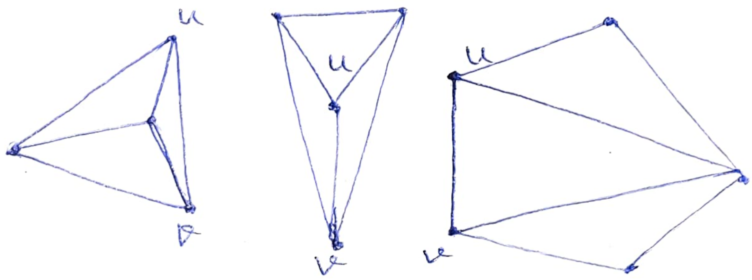
Corollary: (2) For any graph G, $\chi \leq \Delta + 1$

Proof: Suppose ~~$\Delta \geq$~~ $\chi \geq \Delta + 2$. Then, from Corollary 1.
G has at least χ vertices of degree at least $\chi - 1 \geq \Delta + 1$,
which is impossible.

Defn: Let S be a vertex cut of a connected graph G and let the components of $G - S$ have vertex sets $V_1, V_2, \ldots, V_n$. Then the subgraph $G_i = G[V_i \cup S]$ are called the S-components of G. For $S = \{u, v\}$, the S-components are given in the figure. We say that the colourings $G_1, G_2, \ldots, G_n$ agree on S if for every $v \in S$, vertex v is assigned the same colour in each of the colourings.



G



$\{u, v\}$ - Components of G

Theorem (2) In a critical graph, no vertex cut is a clique.

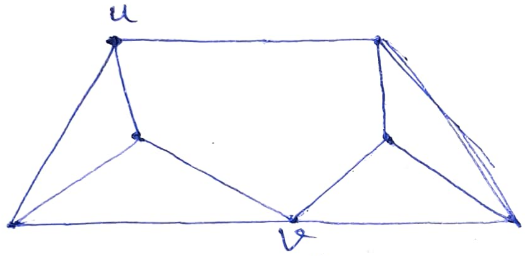
Proof: ~~By Contradiction~~
By Contradiction.
Let G be ~~k be a~~ k-critical graph and suppose that G has a vertex cut S, that is clique. Denote the S-components of G by $G_1, G_2, \ldots, G_n$. Since G is k-critical each G_i is $(k-1)$ colourable. Furthermore, since G is a clique, the vertices in S must receive distinct colours in any $(k-1)$-colouring of G_i. It follows that there are $(k-1)$-colouring of $G_1, G_2, \ldots, G_n$ which agree on S. But these colourings together yield a $(k-1)$-colouring of G, a contradiction.

Hence in a critical graph, no vertex cut is clique.

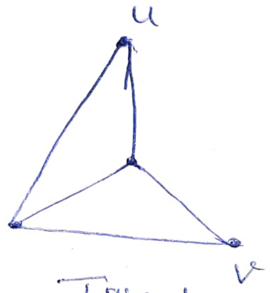
Corollary ③ Every Critical graph is a block

Proof: If v is a cut vertex, then $\{v\}$ is a vertex cut, which is also, trivially a clique. It follows from the theorem ② that no critical graph has a cut vertex; equivalently, every critical graph is a block.

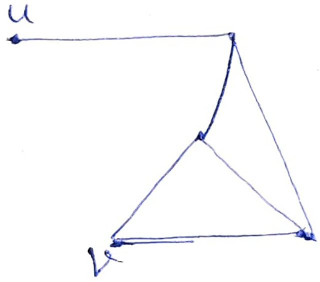
Defn: If a critical graph G has a 2-vertex cut $\{u, v\}$, then u and v cannot be adjacent. We say that a $\{u, v\}$-Component G_i of G is of type 1 if every $(k-1)$-Colouring of G_i assigns the same colour to u and v, and of type 2 if every $(k-1)$-Colouring of G_i assigns different colours to u and v.



graph G



Type 1



Type - 2

Theorem ③ Let G be a k-critical graph with a 2-vertex cut $\{u, v\}$. Then

i) $G = G_1 \cup G_2$, where G_i is a $\{u, v\}$-Component of G of type i $(i = 1, 2)$ and

ii) both $G_1 + uv$ and $G_2 \cdot uv$ are k-critical graphs (where $G_2 \cdot uv$ denotes the graph obtained from G_2 by identifying u and v)

Proof: i) Since G is critical, each $\{u, v\}$-Component of G is $(k-1)$-Colourable. Now, there exist $(k-1)$-colourings of these $\{u, v\}$-Components all of which agree on $\{u, v\}$, since such colourings would together yield a $(k-1)$-colouring of G. Therefore there are two $\{u, v\}$-Components G_1 and G_2 such that no $(k-1)$-Colouring of G_1 agrees with any $(k-1)$-Colouring of G_2. Clearly, one, say G_1, must be of type 1 and the other, G_2 of type 2. Since G_1 and G_2 are different types, the subgraph $G_1 \cup G_2$ of G is not

$(k-1)$- Colourable. Therefore, because G is Critical, ~~we~~
We must have $G = G_1 \cup G_2$

(ii) Set $H_1 = G_1 + uv$. Since G_1 is of type 1, H_1 is k-Chromatic.
We shall prove that H_1 is Critical by showing, for every edge e of H_1, $H_1 - e$ is $(k-1)$- Colourable. This is clearly So if $e = uv$, since then $H_1 - e = G_1$. Let e be some other edge of H_1. In any $(k-1)$- colouring of $G-e$, the vertices u and v must receive different colours, since G_2 is a subgraph of $G-e$. The restriction of such a colouring to the vertices of G is a $(k-1)$- Colouring of $H_1 - e$. Thus $G_1 + uv$ is k-critical.
An analogous arguement shows that $G_2 \cdot uv$ is k-Critical.

<u>Corollary</u> 4 Let G be a k-Critical graph with a 2-vertex cut $\{u, v\}$. Then $d(u) + d(v) \geq 3k-5$

<u>Proof</u>: Let G_1 be $\{u, v\}$- Compenent of type 1 and G_2 be a $\{u, v\}$ Component of type 2. Set $H_1 = G_1 + uv$ and $H_2 = G_2 \cdot uv$
From theorem 3 and the fact that $\delta \geq k-1$, we have

$d_{H_1}(u) \geq k-1$ and $d_{H_1}(v) \geq k-1$, since $\delta \geq k-1$

Therefore, $d_{H_1}(u) + d_{H_1}(v) \geq 2k-2$ and

$d_{H_2}(w) \geq k-1$, where w is the new vertex obtained by identifying u and v

It follows that

$$d_{G_1}(u) + d_{G_1}(v) \geq 2k-2-2$$

$$\Rightarrow \cancel{d\in}\ d_{G_1}(u) + d_{G_1}(v) \geq 2k-4 \rightarrow 1$$

$$d_{G_2}(u) + d_{G_2}(v) \geq d_{H_2}(w)$$

$$\Rightarrow d_{G_2}(u) + d_{G_2}(v) \geq k-1 \rightarrow 2$$

From 1 and 2, we have

$$d_G(u) + d_G(v) \geq 2k-4+k-1$$
$$\geq 3k-5$$

Defn: A graph G is uniquely k-Colourable if any two k-Colourings of G induce the same partition of V.

The cycle C_4 is uniquely 2-Colourable as shown in the Figure