Unit - I - Lagrangian Mechanics

Introduction

* Mechanics ⇒ It is a branch of physics which deals with physical objects in motion and at rest under the influence of internal & external interactions.

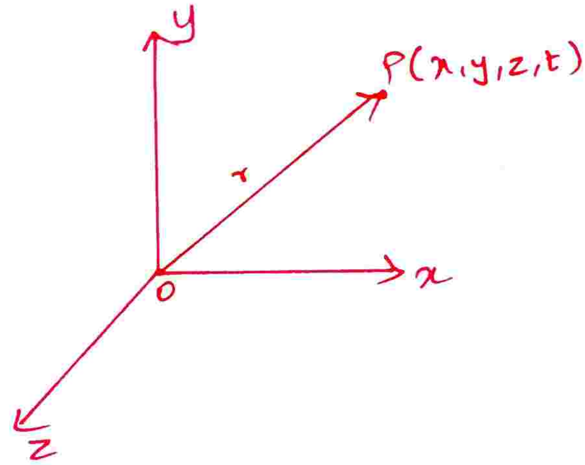
* The mechanics based on Newton's laws of motion and alternatively developed by Lagrange, Hamilton and others is called Classical Mechanics.

* when this mechanics deals with the Newton's laws and their consequences, it may be called as Newtonian or Vectorial mechanics.

* Because in this scheme, the quantities such as force, acceleration, momentum etc., are used which are essentially vectors. The alternative and superior schemes in classical mechanics, developed by D'Alembert, Lagrange, Hamilton and others constitute what is known as analytical mechanics.

Frame of reference

* In order to describe the motion of a particle in space, we need to know its position at different instants of time. This needs the choice of reference body or coordinate system.



\# If we imagine a coordinate system attached to a rigid body and we describe the position of any particle relative to it, then such a coordinate system is called frame of reference.

\# For the location of the objects, the position vectors are drawn from the origin 'O' of the coordinate system. The simplest frame of reference is a cartesian coordinate system.

\# In this system, the position of a particle at any point of its path is gn. by the position vector r, expressed in terms of three coordinates (x, y, z) as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \longrightarrow (1)$$

\# Inorder to know the position of the particle at different instants of time, an observer at the origin with a clock to measure the time t. Thus, we obtain the position vector r of the particle as fn. of time t. i.e.,

$$\vec{r} = \vec{r}(t) \rightarrow (2)$$

\# Thus we obtain the velocity & acceleration as

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \longrightarrow (3)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2r}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k} \rightarrow (4)$$

The position and time recorded together constitute an event, represented by four coordinates (x, y, z, t) & the reference system used for this purpose, may be called as space-time reference system.

Newton's laws of motion

Sir Isaac Newton expressed his ideas regarding the motion of bodies in the form of three laws which are considered as the basic laws of mechanics.

First law (or Law of inertia

A body continues in its state of rest or constant velocity, unless not disturbed by some external influence.

Inertia ⇒ The property of a body that it cannot change its state of rest or uniform motion.

Force ⇒ The influence under which the velocity of a particle changes.

Second law (or law of force

The time-rate of change of momentum is proportional to the impressed force.

i.e., $\vec{F} = \dfrac{d\vec{P}}{dt} \longrightarrow (5)$

Third law (or Law of action & reaction

To every action there is always equal and opposite reaction.

Mechanics of a particle: Conservation Laws

We apply Newtonian mechanics to deduce conservation laws for a particle in motion. These laws tell us under what conditions the mechanical quantities like linear momentum, angular momentum & energy etc., are constant in time.

Conservation of linear momentum

If a force F is acting on a particle of mass m, then according to Newton's second law of motion, we have

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(m\vec{v}) \longrightarrow (6)$$

here $\vec{P} = m\vec{v} \Rightarrow$ linear momentum of the particle.

If the external force, acting on the particle, is zero, then

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(m\vec{v}) = 0$$

(or)

$$\vec{P} = m\vec{v} = \text{const} \longrightarrow (7)$$

Thus in absence of external force, the linear momentum of a particle is conserved. This is the conservation theorem for ~~the~~ a free particle.

Conservation of Angular momentum

The ang. momentum of the particle about a pt. O, denoted by L, is defined as

$$\vec{L} = \vec{r} \times \vec{p} \longrightarrow (1)$$

here $r \rightarrow$ position vector of the particle &
$p \rightarrow$ its linear momentum.

If the force on the particle is F, then the moment of force or torque about O is defined as

$$\vec{N} = \vec{r} \times \vec{F} \rightarrow (2)$$

$$\vec{N} = \vec{r} \times \frac{d}{dt}(m\vec{v}) \rightarrow (3)$$

But $\frac{d}{dt}(\vec{r} \times m\vec{v}) = \frac{d\vec{r}}{dt} \times m\vec{v} + \vec{r} \times \frac{d}{dt}(m\vec{v})$

$$= \underbrace{\vec{v} \times m\vec{v}}_{0} + \vec{r} \times \frac{d}{dt}(m\vec{v})$$

$$\therefore \frac{d}{dt}(\vec{r} \times m\vec{v}) = \vec{r} \times \frac{d}{dt}(m\vec{v}) \rightarrow (4)$$

Sub. eqn (4) in eqn (3), we get

$$\vec{N} = \frac{d}{dt}(\vec{r} \times m\vec{v})$$

(or)

$$\vec{N} = \frac{d\vec{L}}{dt}$$

The time-rate of change of ang. mom. of a particle is equal to the torque acting on it.

If the torque acting on the particle is zero, i.e $N=0$

$$\frac{dL}{dt} = 0 \quad \text{(or)} \quad L = \text{const}$$

$\therefore$ The ang. mom. of a particle is const of motion in absence of external torque. This is the conservation theorem of ang. momentum of a particle.

Conservation of energy

Consider the workdone by the external force F upon the particle in going from pt. 1 to pt. 2.

$$W_{12} = \int_1^2 F\cdot ds \rightarrow (1)$$

$$\int_1^2 F\cdot ds = m\int_1^2 \frac{dv}{dt}\cdot v\,dt = m\left[\frac{v^2}{2}\right]_1^2$$

$$= \frac{m}{2}(v_2^2 - v_1^2)$$

$$\therefore W_{12} = \frac{m}{2}(v_2^2 - v_1^2) \rightarrow (2)$$

The quantity $\frac{mv^2}{2}$ is called the K.E of the particle & denoted by T. Thus the workdone by the force acting on the particle appears equal to the change in the K.E.

i.e., $W_{12} = T_2 - T_1 \rightarrow (3)$

The workdone around a closed ckt is zero. i.e

$$\oint F\cdot ds = 0 \rightarrow (4)$$

For conservative system, the conservative force

$$F = -\nabla V(r) \rightarrow (5)$$

here $V \rightarrow$ the potential or potential energy

$$F\cdot ds = -dv \rightarrow (6)$$

(or $F = -\frac{\partial v}{\partial s}$

For a conservative system, the workdone by the forces in

$$\int_1^2 F\cdot ds = -\int_1^2 dv = -(V_2 - V_1)$$

$$W_{12} = V_1 - V_2 \rightarrow (7)$$

Combining eqn. (3) & (7), we have

$$T_1 + V_1 = T_2 + V_2$$

Thus the sum of K.E + P.E (i.e., total mechanical energy) of a particle remains const in a conservative force field. This is known as the law of conservation of energy.

Mechanics of a system of particles

a) Conservation theorem for a system of particles

If a mechanical system consists of two or more particles. The system of particles will experience two types of forces. They are

(i) The external forces due to agents outside the system.

(ii) the internal forces due to all other particles of the system.

The total force F_i acting on i^{th} particle can be expressed as

$$F_i = F_i^e + \sum_j F_{ji} \rightarrow ①$$

here $F_i^e \rightarrow$ external force

$F_{ji} \rightarrow$ internal force acting on i^{th} particle due to j^{th} particle.

$\sum F_{ji}$ represents the total internal force acting on i^{th} particle due to all other particles in the system.

The eqn of motion for i^{th} particle can be written as

$$\dot{P}_i = F_i^e + \sum F_{ji}$$

here $P_i \rightarrow$ linear momentum of the particle.

Now to consider the system as a whole, we sum over all the particles, eqn ① becomes

$$\sum_i \dot{P}_i = \sum F_i^e + \sum F_{ji} \rightarrow ②$$

Since when $i=j$, $F_{ij} = F_{ji} = 0$

In eqn ② R.H.S, the first term is the total external force & the second term will be vanished.

$\therefore$ eqn ② becomes

$$\sum \dot{P}_i = \sum F_i^e = F^e$$

$$\sum_i \frac{dP_i}{dt} = F^e$$

$$\frac{d}{dt} \sum (P_i) = F^e$$

$$\frac{dP}{dt} = F^e$$

If $F^e = 0$ $\frac{dP}{dt} = 0$ $\therefore P = \text{const}$

If the sum of external forces acting on the system of particle is zero, the total linear momentum of the system is const. or conserved.

Conservation theorem for angular momentum

The ang. momentum of the ith particle of the system is

$$L_i = r_i \times P_i \rightarrow ①$$

where $r_i \rightarrow$ the radius vector of ith particle from pt. O. &

$P_i \rightarrow$ its linear momentum

But $P_i = m_i V_i$

where $V_i \rightarrow$ velocity of ith particle

$$L_i = r_i \times m_i V_i \rightarrow ②$$

The torque (or) moment of force acting on ith particle is gn. by $N_i = r_i \times F_i$

where $F_i \rightarrow$ the force acting on ith particle

But $F_i = \frac{dP_i}{dt} = \frac{d}{dt}(m_i v_i) \rightarrow ③$

$$N_i = r_i \times \frac{d}{dt} m_i v_i \rightarrow (4)$$

But $$\frac{d}{dt}(r_i \times m_i v_i) = r_i \times \frac{d}{dt}(m_i v_i) \rightarrow (5)$$

sub. eqn (5) in eqn (4), we have

$$N_i = \frac{d}{dt}(r_i \times m_i v_i)$$

$$N_i = \frac{d}{dt}(r_i \times P_i)$$

$$N_i = \frac{dL_i}{dt}$$

The rate of change of ang. mom is equal to the torque acting on the particle.

If $N_i = 0$, $\frac{dL_i}{dt} = 0$ $L_i = \text{const}$

If the total torque acting on the system of particles is zero, the total ang. momentum of the system is conserved.

conservation theorem for energy

If W_i is the workdone by external force F_i acting on i^{th} particle is displacing it from position 1 to 2, then

$$W_i = \int_1^2 F_i \cdot dr_i = \int_1^2 m_i \ddot{r}_i \cdot dr_i$$

$$= m_i \left(\frac{V_i^2}{2}\right)_1^2 = \frac{m_i}{2}(V_2^2 - V_1^2)$$

$$W_{12} = T_2 - T_1 \rightarrow (1)$$

here $T_i \Rightarrow$ the K.E of the i^{th} particle.

For a conservative system, the force F_i is expressed as the gradient of some scalar function. i.e $F_i = -\nabla V_i$

where $V_i \Rightarrow$ the P.E of the i^{th} particle

$$\nabla V_i = \frac{\partial V_i}{\partial r_i}$$

$$F_i = -\frac{\partial V_i}{\partial r_i}$$

Workdone by the external force F_i is displacing it from 1 to 2 is gn by

$$W_i = \int_1^2 F_i \cdot dr_i = -\int_1^2 \frac{\partial V_i}{\partial r_i} \cdot dr_i = -(V_i)_1^2$$

$$W_{12} = V_1 - V_2 \rightarrow ②$$

combining eqn ① & ②, we get

$$T_2 - T_1 = V_1 - V_2$$

(or)

$$T_1 + V_1 = T_2 + V_2 = E \Rightarrow \text{Total energy of the system.}$$

If the forces acting on the system of particles are conservative the total energy of the system of particles which is the sum of total K.E & the total P.E of the system is conserved.

constraints

* The limitations or geometrical restrictions on the motion of a particle or system of particles generally known as constraints.

Eg: The motion of the gas molecules within the container which is restricted by the walls since gas molecules can move only inside the container.

* constraints are classified into 2 types. They are

 (i) **Holonomic**

 (ii) **Non-holonomic**

* If the conditions of constraint can be expressed as eqns connecting the coordinates of the particles having the form

$$f(r_1, r_2, r_3 \ldots r_n, t) = 0$$

then the constraints said to be holonomic. The constraints cannot be expressed this eqn is called non-holonomic.

* constraints are further classified into 2 types.

If the eqn of constraints contain the time as an explicit variable is called **Rheonomous.**

If the eqn of constraints does not contain the time variable is called **Scleronomous.**

Degrees of freedom

* The minimum no. of independent variables or coordinates required to specify the position of a dynamical system, consisting of one or more particles, is called the no. of degrees of freedom.

Eg:- (i) The motion of a particle, moving freely in space, can be described by a set of three coordinates (x, y, z) and hence the no. of degrees of freedom, possessed by the particle, is three.

(ii) A system of two particles, moving freely in space, requires to set of three coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) i.e., six coordinates to specify its position. Thus the system has six degrees of freedom.

Principle of virtual work

* The virtual displacement of i^{th} particle of a system of N particles is denoted by δr_i

$$\delta r_i = \delta r_i (q_1, q_2, \dots q_n)$$

* Suppose the system is in equilibrium, then the total force on any particle is zero. i.e $F_i = 0$

* The virtual work of the force F_i in the virtual displacement δr_i will also be zero.

i.e., $$W_i = \sum_{i=1}^{N} F_i \cdot \delta r_i = 0 \rightarrow (1)$$

* The principle of virtual work which states that the workdone is zero.

The total force F_i on the i^{th} particle is written as

$$F_i = F_i^a + f_i \rightarrow (2)$$

here $F_i^a \rightarrow$ applied force

$f_i \rightarrow$ constrained force

Sub. eqn ② in eqn ①, we get

$$\sum_{i=1}^{N} F_i^a \cdot \delta r_i + \underbrace{\sum_{i=1}^{N} f_i \cdot \delta r_i}_{0} = 0 \rightarrow ③$$

$$\boxed{\sum_{i=1}^{N} F_i^a \cdot \delta r_i = 0} \rightarrow \text{principle of work}$$

D' Alembert's principle

From Newton's second law of motion

$$F_i = \frac{dP_i}{dt} = \dot{P}_i$$

(or)

$$F_i - \dot{P}_i = 0$$

Virtual workdone can be written as

$$\sum_{i=1}^{N} (F_i - \dot{P}_i) \cdot \delta r_i = 0$$

But $F_i = F_i^a + f_i$ then

$$\sum_{i=1}^{N} (F_i^a - \dot{P}_i) \cdot \delta r_i + \underbrace{\sum_{i=1}^{N} (f_i \cancel{-\dot{P}_i}) \cdot \delta r_i}_{} = 0$$

This is known as D' Alembert's principle.

Since the forces of constraints donot appear in the eqn & hence we can drop the superscript.

The D'Alembert's principle may be written as

$$\sum_{i=1}^{N} (F_i - \dot{P}_i) \cdot \delta r_i = 0$$

Lagrange's equation.

Consider a system of N particles. The transformation eqns for the position vectors of the particles are

$$r_i = r_i(q_1, q_2, \dots q_n, t) \rightarrow ①$$

differentiating eqn ① w.r.to t, we have

$$\frac{dr_i}{dt} = \frac{\partial r_1}{\partial q_1} \cdot \frac{\partial q_1}{\partial t} + \frac{\partial r_2}{\partial q_2} \cdot \frac{\partial q_2}{\partial t} + \dots\dots + \frac{\partial r_i}{\partial q_n}\frac{\partial q_n}{\partial t} + \frac{\partial r_i}{\partial t}$$

$$v_i = \dot{r}_i = \sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t} \rightarrow ②$$

↳ gen. velocity

The virtual displacement is gn. by

$$\delta r_i = \sum_j \frac{\partial r_i}{\partial q_j} \delta q_j$$

According to D' Alembert's principle

$$\sum_{i=1}^{N} (F_i - \dot{P}_i) \cdot \delta r_i = 0 \rightarrow ③$$

where $\sum_i F \cdot \delta r_i = \sum_{i,j} F_i \frac{\partial r_i}{\partial q_j} \delta q_j = \sum_j Q_j \cdot \delta q_j \rightarrow ④$

here $Q_j \rightarrow$ generalized force, defined as

$$Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j}$$

2nd term in ③ $\qquad$ $ma = m\frac{dv}{dt} = \frac{d}{dt}(mv)$

$$\sum_i \dot{P}_i \cdot \delta r_i = \sum m_i \ddot{r}_i \cdot \delta r_i \rightarrow ⑤$$

$$\sum_{i,j} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} \delta q_j \rightarrow ⑥$$

consider the relation

$$\sum_i m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{r}_i \cdot \frac{\partial r_i}{\partial q_j}\right) - m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)\right] \rightarrow ⑦$$

In eqn ⑦ Second term becomes

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \frac{\partial \dot{r}_i}{\partial q_j} = \frac{\partial v_i}{\partial q_j} \rightarrow ⑧$$

diff. eqn ② w.r.to $\dot{q}_j$, we get

$$\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j} \rightarrow ⑨$$

∴ eqn ⑦ becomes

$$\sum_i m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i v_i \cdot \overset{⑨}{\frac{\partial v_i}{\partial \dot{q}_j}}\right) - m_i v_i \overset{⑧}{\frac{\partial v_i}{\partial q_j}}\right] \rightarrow ⑩$$

Substitute eqn ⑩ in eqn ③ & multiply by −ve, we get

$$\sum_j \left\{\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \tfrac{1}{2} m_i v_i^2\right)\right] - \frac{\partial}{\partial q_j}\left(\sum_i \tfrac{1}{2} m_i v_i^2\right) - Q_j\right\} \cdot \delta q_j = 0$$

But $T = \sum_i \frac{1}{2} m_i v_i^2$

$$\sum \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j \right\} \cdot \delta q_j = 0$$

(or)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \rightarrow ⑪$$

For conservative systems, the force is defined as

$$F_i = -\nabla V_i$$

Then the gen. forces can be written as

$$Q_j = \sum_i F_i \cdot \frac{\partial r_i}{\partial q_j} = -\sum_i \nabla V_i \cdot \frac{\partial r_i}{\partial q_j} = -\frac{\partial V_i}{\partial r_i} \cdot \frac{\partial r_i}{\partial q_j}$$

$$Q_j = -\frac{\partial V_i}{\partial q_j} \rightarrow ⑫$$

eqn ⑪ can be rewritten as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0 \rightarrow ⑬$$

The eqns of motion in the form (eqn 13) are not necessarily restricted to conservative systems; only if V is not an explicit fn. of time is the system conservative. As here defined, the potl V does not depend on the generalized velocities. Hence we can include a term in V in the partial derivative w. r. to $\dot{q}_j$.

$$\frac{d}{dt}\left(\frac{\partial(T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0 \rightarrow (14)$$

The new fn. the Lagrangian L, defined as

$$L = T - V$$

The eqn (14) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \left(\frac{\partial L}{\partial q_j}\right) = 0 \rightarrow (15)$$

This eqn is known as Lagrange's eqns of motion.

Hamilton's principle & Lagrange's equation

We have used the D'Alembert's principle to deduce Lagrange's eqns. This principle uses the idea of virtual work & Newton's II Law of motion. These Lagrange's eqns can be derived by an entirely different way, namely Hamilton's Variational principle.

Hamilton's principle

This pple states that for a conservative holonomic system, its motion from time t_1 to t_2 is such that line integral

$$I = \int_{t_1}^{t_2} L\,dt \longrightarrow (1)$$

with $L = T - V$ has stationary value for the correct path of motion.

The quantity I is known as Hamilton's pple. function. The pple may be expressed as

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \longrightarrow (2)$$

where $\delta \rightarrow$ Variation symbol.

Lagrange's equation from Hamilton's principle

The Lagrangian L is a fn of generalised co-ordinates q_k's & generalised velocities $\dot{q}_k$'s and time t. ie

$$L = L(q_1, q_2 \dots q_k, \dot{q}_1, \dots \dot{q}_k, t)$$

If the Lagrangian does not depend on time t explicitly, then the variation δL can be written as

$$\delta L = \sum_k \frac{\partial L}{\partial q_k} \delta q_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \quad \text{--- (3)}$$

Integrating both sides from $t = t_1$ to $t = t_2$, we get

$$\int_{t_1}^{t_2} \delta L \, dt = \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k \, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \, dt$$

From Hamilton's pple

$$\delta \int_{t_1}^{t_2} L \, dt = 0$$

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k \, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \, dt = 0 \quad \text{--- (4)}$$

where $\delta \dot{q}_k = \frac{d}{dt}(\delta q_k)$

Integrating by parts the 2nd term on the L.H.S

$$\boxed{\int u \, dv = uv - \int v \, du}$$

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \, dt = \sum_k \underbrace{\left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k\right]_{t_1}^{t_2}}_{0} - \int_{t_1}^{t_2} \delta q_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) dt$$

$$= -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \delta q_k \, dt \quad \text{--- (5)}$$

Sub ⑤ in ④ we have

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k \, dt + \int_{t_1}^{t_2} \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \delta q_k \, dt = 0$$

Multiply by (-ve)

$$\sum_k \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \left(\frac{\partial L}{\partial q_k}\right) \right] \delta q_k \, dt = 0$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \left(\frac{\partial L}{\partial q_k}\right) = 0}$$

This eqn is called Lagrange's eqn of motion

Simple applications of the Lagrangian formulation

a) Motion of one particle using cartesian coordinates

The generalized forces needed in eqn ⑪ $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$ are obviously F_x, F_y & F_z. Then

$$T = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$\left[\because \tfrac{1}{2} m v^2 = \tfrac{1}{2} m \dot{r}^2 = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right]$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0$$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial T}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

and the eqns of motion are

$$\frac{d}{dt}(m\dot{x}) = F_x$$

$$\frac{d}{dt}(m\dot{y}) = F_y$$

$$\frac{d}{dt}(m\dot{z}) = F_z$$

$\because F = \frac{dp}{dt} = \frac{d}{dt} m\dot{r} = \frac{d}{}$

$F_x = \frac{d}{dt} m\dot{x}$

$\boxed{F_x = m \frac{d\dot{x}}{dt}}$

b) Motion of one particle using plane polar coordinates

Here we express T in terms of $\dot{r}$ & $\dot{\theta}$. The eqns of transformation, i.e., eqns ($r = r(q_1, q_2 \dots q_n, t)$), in this case are simply

$$x = r\cos\theta \quad \rightarrow ①$$

$$y = r\sin\theta \quad \rightarrow ②$$

The velocities are given by

$$\dot{x} = \frac{dx}{dt} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \quad \rightarrow ③$$

$$\dot{y} = \frac{dy}{dt} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \quad \rightarrow ④$$

The kinetic energy $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$ then reduces to

$$T = \frac{1}{2}m\left[\dot{r}^2+(r\dot{\theta})^2\right]$$

From eqns ③ & ④

$$\dot{x}^2 = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2$$

$$= \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta \quad \rightarrow ⑤$$

$$\dot{y}^2 = (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2$$

$$= \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta \quad \rightarrow ⑥$$

Adding eqn ⑤ & ⑥

$$(\dot{x}^2+\dot{y}^2) = \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\dot{\theta}\cancel{\cos\theta}\sin\theta + \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + 2r\dot{r}\dot{\theta}\cancel{\cos\theta}\sin\theta$$

$$= \dot{r}^2(\cos^2\theta+\sin^2\theta) + r^2\dot{\theta}^2(\sin^2\theta+\cos^2\theta)$$

$$= \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + (r\dot{\theta})^2$$

$$\therefore (\dot{x}^2+\dot{y}^2) = \dot{r}^2 + (r\dot{\theta})^2$$

$$T = \frac{1}{2}m(\dot{r}^2+(r\dot{\theta})^2] \rightarrow ⑦$$

There are two generalized coordinates, & therefore two Lagrange eqns. The derivatives occuring in the r equation are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_j$$

From ⑦

$$\frac{\partial T}{\partial \dot{r}} = \frac{m}{\cancel{2}}\cdot\cancel{2}\dot{r} = m\dot{r}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) = \frac{d(m\dot{r})}{dt} = m\ddot{r}$$

$$\frac{\partial T}{\partial r} = \frac{m}{\cancel{2}} \cdot \cancel{2} r \dot{\theta}^2 = m r \dot{\theta}^2$$

& the equation becomes

$$m\ddot{r} - m r \dot{\theta}^2 = F_r$$

For the θ equation, we have the derivatives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

$$\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{m}{2} \cdot 2r^2\dot{\theta} = m r^2 \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}(m r^2 \dot{\theta}) = m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta}$$

$$\frac{\partial T}{\partial \theta} = 0$$

So that the eqn. becomes.

$$\frac{d}{dt}(m r^2 \dot{\theta}) = m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta} = r.F_\theta$$

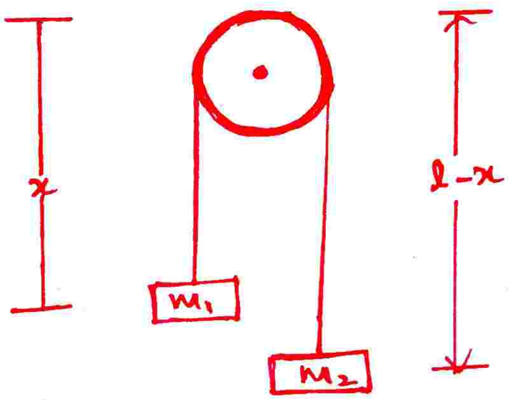
The L.H.S of the eqn is the time derivative of the angular momentum, & the R.H.S is exactly the applied torque. So that we have simply rederived the torque eqn, where $L = m r^2 \dot{\theta}$ & $N = r F_\theta$.

Atwood's machine

* The Atwood's machine is an example of a conservative system with holonomic constraint. The pulley is small, massless & frictionless.

* Let the two masses be m_1 & m_2 which are connected by an inextensible string of length l.

* Suppose x be the variable vertical distance from the pulley to the mass m_1. Then mass m_2 will be at a distance $l-x$ from the pulley. (Dia)



Diag: Atwood's machine

There is only one independent coordinate x. The velocities of the two masses are

$$V_1 = \frac{dx}{dt} = \dot{x}$$

&

$$V_2 = \frac{d(l-x)}{dt} = -\dot{x}$$

$$\therefore \quad T = \tfrac{1}{2} m_1 \dot{x}^2 + \tfrac{1}{2} m_2 \dot{x}^2 = \tfrac{1}{2}(m_1+m_2)\dot{x}^2$$

Potential energy of the system with reference to the pulley is

$$V = -m_1 g x - m_2 g(l-x)$$

Thus the Lagrangian is

$$L = T - V = \frac{1}{2}(m_1+m_2)\dot{x}^2 + m_1 g x + m_2 g(l-x)$$

$$\frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} \quad \& \quad \frac{\partial L}{\partial x} = (m_1-m_2)g$$

Here the generalized coordinate is $q = x$. Now the Lagrangian is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}(m_1+m_2)\dot{x} - (m_1-m_2)g = 0$$

$$(m_1+m_2)\ddot{x} - (m_1-m_2)g = 0$$

$$(m_1+m_2)\ddot{x} = (m_1-m_2)g$$

$$\ddot{x} = \frac{(m_1-m_2)g}{m_1+m_2}$$

which is the desired eqn of motion.

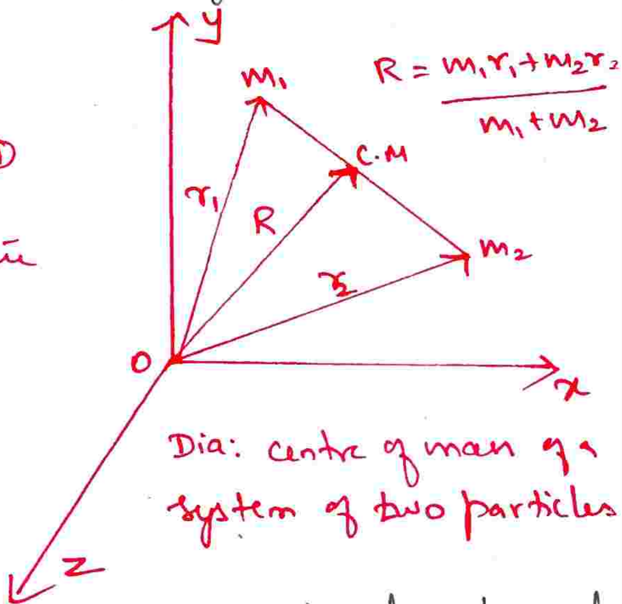
If $m_1 > m_2$, the mass m_1 descends with constant acceleration and if $m_1 < m_2$, the mass m_1 ascends with const. ~~con~~ acceleration. It is to be noted that the tension in the rope, the force of constraint, is not seen anywhere in the Lagrangian formulation.

Centre of mass

The centre of mass R of the system is gn by

$$R = \frac{\sum_i m_i r_i}{\sum_i m_i} = \frac{\sum_i m_i r_i}{M} \rightarrow (1)$$

here $\sum_i m_i = M \rightarrow$ total mass of the system



Dia: centre of mass of a system of two particles

From (1)

$$F^e = M \frac{d^2 R}{dt^2} = M a_{cm}$$

Thus the acceleration of the centre of mass is due to only the external forces and is gn by Newton's second law of motion.

$\therefore$ The centre of mass of a system of particles moves as if it were a particle of mass equal to the total mass of the system subjected to the external forces applied on the system.

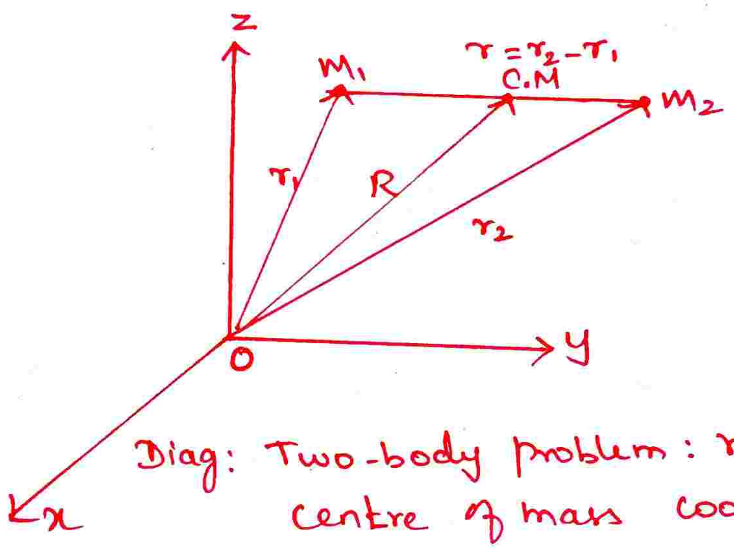
II. The central force problem

Reduction of two-body central force problem to the equivalent one-body problem.

Consider the motion of two bodies under a mutual central force as an application of Lagrangian formulation.

m_1 & m_2 → masses of a system of two particles

r_1 & r_2 → position vectors with origin 'O' of masses m_1 & m_2.



Diag: Two-body problem: relative & centre of mass coordinates

The vector distance of m_2 relative to m_1 is

$$r = r_2 - r_1 \quad \rightarrow ①$$

The two masses are interacting via central force for which the P.E for the system $V(r)$ is a fn of scalar distance r only.

The Lagrangian for the system is

$$L = T - V = \tfrac{1}{2} m_1 \dot{r}_1^2 + \tfrac{1}{2} m_2 \dot{r}_2^2 - V(r) \quad \rightarrow ②$$

This system of two particles has six degrees of freedom & hence six independent generalized coordinates are required to describe the state of the system.

Instead of r_1 & r_2 (6) coordinates, we can choose the three components of the position vector of the centre of mass R, and three components of the relative vector $r = r_2 - r_1$.

The position vector of the centre of mass is defined by

$$R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \longrightarrow ③$$

Solving ① & ③, we get

From ①

$$R = \frac{m_1 r_1 + m_2 (r + r_1)}{m_1 + m_2}$$

$$= \frac{m_1 r_1 + m_2 r + m_2 r_1}{m_1 + m_2}$$

$$= \frac{(m_1 + m_2) r_1 + m_2 r}{m_1 + m_2}$$

$$R = r_1 + \frac{m_2}{m_1 + m_2} r$$

$$\left.\begin{aligned} r_1 &= R - \frac{m_2}{m_1 + m_2} r \\ \text{lly} \quad r_2 &= R + \frac{m_1 r}{m_1 + m_2} \end{aligned}\right\} \rightarrow ④$$

$$\left.\begin{aligned} \therefore \quad \dot{r}_1 &= \dot{R} - \frac{m_2}{m_1 + m_2} \dot{r} \\ \& \quad \dot{r}_2 &= \dot{R} + \frac{m_2}{m_1 + m_2} \dot{r} \end{aligned}\right\} \rightarrow ⑤$$

Hence $$L = \tfrac{1}{2} m_1 \left(\dot{R} - \frac{m_2 \dot{r}}{m_1+m_2} \right)^2 + \tfrac{1}{2} m_2 \left(\dot{R} + \frac{m_1 \dot{r}}{m_1+m_2} \right)^2 - V(r)$$

(or)

$$= \tfrac{1}{2} m_1 \left[\dot{R}^2 + \frac{m_2^2 \dot{r}^2}{(m_1+m_2)^2} - \frac{2 m_2 \dot{R} \dot{r}}{m_1+m_2} \right] + \tfrac{1}{2} m_2 \left[\dot{R}^2 + \frac{m_1^2 \dot{r}^2}{(m_1+m_2)^2} + \frac{2 m_1 \dot{R} \dot{r}}{m_1+m_2} \right] - V(r)$$

$$= \tfrac{1}{2} m_1 \dot{R}^2 + \tfrac{1}{2} \frac{m_1 m_2^2 \dot{r}^2}{(m_1+m_2)^2} - \cancel{\frac{m_1 m_2 \dot{R}\dot{r}}{m_1+m_2}} + \tfrac{1}{2} m_2 \dot{R}^2 + \tfrac{1}{2} \frac{m_1^2 m_2 \dot{r}^2}{(m_1+m_2)^2} + \cancel{\frac{m_1 m_2 \dot{R}\dot{r}}{m_1+m_2}} - V(r)$$

$$= \tfrac{1}{2}(m_1+m_2) \dot{R}^2 + \tfrac{1}{2} \frac{m_1 m_2 (m_1+m_2)}{(m_1+m_2)^2} \dot{r}^2 - V(r)$$

$$L = \tfrac{1}{2}(m_1+m_2) \dot{R}^2 + \tfrac{1}{2} \frac{m_1 m_2}{m_1+m_2} \dot{r}^2 - V(r) \rightarrow (6)$$

It is seen that the three coordinates R are cyclic, so that the centre of mass is either at rest or moving uniformly. Obviously the Lagrange's eqns of motion for the three gen. coordinates r will not contain the terms R & $\dot{R}$. We drop the 1st term from the Lagrangian.

$$L = \tfrac{1}{2} \mu \dot{r}^2 - V(r) \rightarrow (7)$$

here $\mu = \dfrac{m_1 m_2}{m_1+m_2}$ is called the reduced mass of the two-particle system. $\rightarrow (8)$

Eqn (8) can be written in the form

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \quad -(9)$$

Thus, the central force motion of two bodies about their centre of mass can always be reduced to an equivalent one-body problem.

central force & Motion in a plane

* If a force acts on a particle in such a way that is always directed towards or away from a fixed centre and its magnitude depends only upon the distance (r) from the centre, then this force is called central force. Thus a central force is represented by

$$\vec{F} = f(r)\hat{r} = f(r)\frac{\vec{r}}{r} \quad \text{—} ① \qquad \because \hat{r} = \frac{\vec{r}}{r}$$

here $f(r) \rightarrow$ a fn. of distance r only

$\hat{r} \rightarrow$ a unit vector along r from the fixed centre.

* The force may be attractive or repulsive, if $f(r) < 0$ (or) $f(r) > 0$ respectively.

* A central force is always a conservative force & if $V(r)$ is the P.E, then

$$f(r) = -\frac{\partial V}{\partial r} \quad (\text{or}) \quad F = -\frac{\partial V}{\partial r}\frac{\vec{r}}{r} \quad \text{—} ② \quad \text{from } ①$$

* The P.E for central force depends only on the distance r & hence the system possesses spherical symmetry. Thus any rotation about a fixed axis will not have any effect on the solution & hence an angle coordinate for rotation about a fixed axis must be cyclic.

From the conservation of angular momentum of the system

i.e., $\vec{L} = \vec{r} \times \vec{P} = \text{const} \longrightarrow ③$

↳ linear momentum

Taking dot product with r in eqn ③, we have

$$\vec{r}.\vec{L} = \vec{r}.(\vec{r} \times \vec{P}) \qquad (\because \vec{A} \times \vec{A} = 0)$$

$$= (\vec{r} \times \vec{r})\cdot\vec{P} = 0 \longrightarrow ④$$

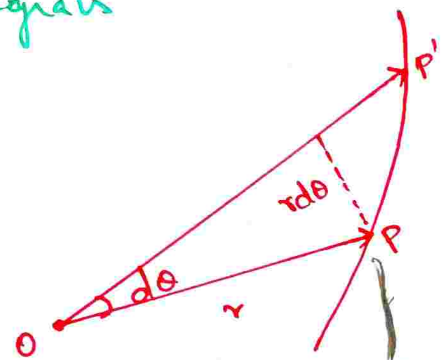
also called dot triple product

Since in a scalar triple product the position of dot & cross are interchangeable & $\vec{r}\cdot\times\vec{r} = 0$

$A\cdot B = \cos\theta$
$A\times B = AB\sin\theta$

$\therefore$ The position vector $\vec{r}$ is always $\perp^r$ to the constant L vector. This means that the motion of the particle under central force takes place in a plane & we can describe the instantaneous position of the particle in plane polar coordinates r & θ.

Equations of motion under central force & First Integrals



Diag: Area swept out by the radius vector in infinitesimal small time dt

* Consider a particle of mass m moving about a fixed centre of force O.

* The particle is moving under a central force

* $\vec{F} = f(r)\dfrac{\vec{r}}{r} = -\dfrac{\partial V}{\partial r}\dfrac{\vec{r}}{r}$

here $V(r) \rightarrow$ P.E

Using polar coordinates (r, θ), the Lagrangian for the system can be written as

$$L = T - V = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \rightarrow (1)$$

In eqn (1), the Lagrangian L is independent of θ coordinate (i.e., $\frac{\partial L}{\partial \theta} = 0$) & hence θ is the cyclic coordinate. The canonical momentum P_θ corresponding to the coordinate θ is given by

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \rightarrow (2)$$

$\hookrightarrow$ ang. momentum

One of the two equations of motion (Lagrange's eqn for θ coordinate) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad (\text{or}) \quad \frac{d}{dt}(mr^2\dot{\theta}) = 0 \rightarrow (3)$$

Integration of this eqn gives one of the first integral of motion.

i.e., $$mr^2\dot{\theta} = l \longrightarrow (4)$$

here $l \rightarrow$ the const magnitude of the ang. mom & is conserved.

Since m is a const, we obtain from eqn (3)

$$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0 \text{ (or) } \frac{1}{2}r^2\dot{\theta} = \text{constant} \longrightarrow (5)$$

The factor $\frac{1}{2}$ is inserted because $\frac{1}{2}r^2\dot{\theta}$ is the areal velocity - the area swept out by the radius vector per unit time.

The differential area swept out in time dt is

$$dA = \frac{1}{2}r(rd\theta)$$

& hence

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$$

The conservation of ang. mom. is thus equivalent to the areal velocity is constant. Here we have the proof of the kepler's second law of planetary motion: The radius vector sweeps out equal areas in equal times. It should be emphasized however that the conservation of the areal velocity is a general property of central force motion & is not restricted to an inverse-square law of force.

The remaining Lagrange eqn, for the coordinate r, is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \left(\frac{\partial L}{\partial r}\right) = 0 \longrightarrow (6)$$

From eqn (1) $$\left(\frac{\partial L}{\partial \dot{r}}\right) = m\dot{r} \quad \& \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{d}{dt}(m\dot{r}) = m\ddot{r}$$

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \rightarrow (7)$$

Designating the value of the force along r, $-\frac{\partial V}{\partial r}$, by f(r) the eqn can be rewritten as

$$m\ddot{r} - mr\dot{\theta}^2 = f(r) \rightarrow (8)$$

From eqn (4), $\dot{\theta}$ can be eliminated from the eqn of motion, yielding a second order diff. eqn involving r only:

$$\dot{\theta} = \frac{l}{mr^2}$$

eqn (8) becomes

$$m\ddot{r} - \cancel{mr}\frac{l^2}{m^{\cancel{2}}r^4} = f(r)$$

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r) \rightarrow (9)$$

There is another first integral of motion available, namely the total energy, since the forces are conservative. On the basis of the general energy conservation thm, we can state that the const. of the motion is

$$E = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \rightarrow (10)$$

where E is the energy of the system. Alternatively, this first integral could be derived again directly from the eqns of motion eqn (5) & (9). The latter can be written as

$$\frac{l^2}{mr^3} = \frac{l^2}{m} r^{-3}$$

$$m\ddot{r} = \frac{l^2}{mr^3} - \frac{\partial V}{\partial r}$$

(or)

$$= -\frac{1}{2}\frac{\partial}{\partial r}\left[\frac{l^2}{mr^2}\right] - \frac{\partial V}{\partial r}$$

$$\because \frac{\partial}{\partial r}(r^{-2}) = -2r^{-3}$$

$$r^{-3} = -\frac{1}{2}\frac{\partial}{\partial r}(r^{-2})$$

$$\frac{l^2}{m}r^{-3} = -\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{l^2}{mr^2}\right)$$

$$m\ddot{r} = -\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{l^2}{mr^2} + V\right) \rightarrow (11)$$

Multiplying both sides of eqn (11) by $\dot{r}$, we get

$$m\ddot{r}\dot{r} = -\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{l^2}{mr^2} + V\right)\dot{r} \text{ (or) } \frac{\partial r}{\partial t}$$

(or) $$\frac{d}{dt}\left(\frac{1}{2} m\dot{r}^2\right) = -\frac{d}{dt}\left(\frac{l^2}{2mr^2} + V\right)$$

(or)

$$\frac{d}{dt}\left(\frac{1}{2} m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} + V\right) = 0 \rightarrow (12)$$

Integrating it, we get

$$\frac{1}{2} m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} + V(r) = E\,(\text{const}) \rightarrow (13)$$

eqn (13) is the stmt of the conservation of ~~Total~~ Total Energy.

eqn (4) & (13) are known as the first integrals of motion.

Differential equation for an orbit

In case of central force, we deduce the eqn of the orbit whose solution give us the radial distance (r) as a fun. of $\dot{\theta}$.

The eqn of motion for a particle of reduced mass m, moving under central force, can be written as

$$m\ddot{r} - \frac{l^2}{mr^3} = f(r) \longrightarrow (1)$$

Now,

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt}$$

$$= \frac{dr}{d\theta} \cdot \dot{\theta}$$

$$\dot{r} = \frac{dr}{d\theta} \cdot \frac{l}{mr^2} \longrightarrow (2) \qquad \because \dot{\theta} = \frac{l}{mr^2}$$

and

$$\ddot{r} = \frac{d}{dt}\left(\frac{l}{mr^2} \frac{dr}{d\theta}\right)$$

$$= \frac{d}{d\theta}\left(\frac{l}{mr^2} \frac{dr}{d\theta}\right)\frac{d\theta}{dt}$$

$$= \frac{d}{d\theta}\left(\frac{l}{mr^2} \frac{dr}{d\theta}\right)\frac{l}{mr^2}$$

$$\ddot{r} = \frac{l}{mr^2} \frac{d}{d\theta}\left(\frac{l}{mr^2} \frac{dr}{d\theta}\right)$$

Let $u = \frac{1}{r}$, then $\frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta}$

$$\ddot{r} = -\frac{l^2u^2}{m^2}\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)$$

$$= -\frac{l^2u^2}{m^2}\frac{d^2u}{d\theta^2} \longrightarrow (3)$$

Sub. eqn (3) in eqn (1), we get

$$-\frac{l^2u^2}{m}\frac{d^2u}{d\theta^2} - \frac{l^2u^3}{m} = f(1/u)$$

(or)

$$\frac{l^2u^2}{m}\left(\frac{d^2u}{d\theta^2}\right) + \frac{l^2u^3}{m} = -f(1/u)$$

$$\frac{l^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = -f(1/u) \longrightarrow (4)$$

eqn (4) is called as the differential equation of an orbit

Inverse Square law of force

* Gravitational and coulomb force b/w two particles are the most important examples of central force.
* The force $f(r)$ is expressed as

$$f(r) = \frac{Gm_1m_2}{r^2} \rightarrow \text{(Newton's law of Gravitation)} \longrightarrow (1)$$

&

$$f(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2} \rightarrow \text{(coulomb's law)} \longrightarrow (2)$$

The general force law, governing eqn (1) & (2), is the inverse square law of force, gn by

7

$$f(r) = -\frac{k}{r^2} \longrightarrow (3)$$

If V is the potential, then

$$f(r) = -\frac{\partial V}{\partial r} = -\frac{k}{r^2}$$

If we integrating the above eqn, we get

$$V = -\frac{k}{r}$$

here the integration const is taken to be zero by assuming $V(r)=0$ at infinite separation $(r=\infty)$

Kepler's laws of planetary motion and their deduction

Kepler's laws of planetary motion are as follows

(i) The law of elliptical orbits

Every planet moves in an elliptical orbit around the sun, the sun being at one of the foci.

(ii) The law of areas

The radius vector, drawn from the sun to a planet sweeps out equal areas in equal time. i.e., the areal velocity of the radius vector is const.

(iii) The harmonic law

The square of the period of revolution of the planet around the sun is proportional to the cube of the semi-major axis of the ellipse.

The planets move around the sun under the influence of gravitational force which is an inverse square law of force. Hence we deduce kepler's laws on the basis of inverse square law of force.

The inverse square law is the most important of all the central force laws. For this case, the force + potl can be written as

$$f(r) = -\frac{k}{r^2}$$

$$V = -\frac{k}{r}$$

$V = -\frac{k}{r}$

$\frac{\partial V}{\partial r} = \frac{k}{r^2} \Rightarrow -\frac{1}{r} = \frac{1}{r^2}$

$f(r) = -\frac{\partial V}{\partial r} = -\frac{k}{r^2}$

For $u = 1/r$, the inverse square law force is given by $f(1/u) = -ku^2$

According to the differential eqn of an orbit

$$\frac{d^2u}{d\theta^2} + u = \frac{m}{l^2 \cancel{u^2}} k\cancel{u^2}$$

$\because f(r) = -ku^2$

(or)

$$\frac{d^2u}{d\theta^2} + u - \frac{mk}{l^2} = 0 \rightarrow (1)$$

Let $x = u - \frac{mk}{l^2}$, then $\frac{d^2x}{d\theta^2} = \frac{d^2u}{d\theta^2}$

$$\frac{d^2x}{d\theta^2} + x = 0 \rightarrow (2)$$

$\therefore$ The solution is gn by

$$x = A\cos(\theta - \theta') \rightarrow (3)$$

here A + θ $\rightarrow$ the consts of integration

Since $x = u - \frac{mk}{l^2}$ + $u = 1/r$ $\rightarrow (4)$

Comparing eqn (3) + (4), we have

$$\frac{1}{r} - \frac{mk}{l^2} = A\cos(\theta - \theta')$$

$$\frac{1}{r} = \frac{mk}{l^2} + A\cos(\theta - \theta') \rightarrow (5)$$

Multiplying by $\frac{l^2}{mk}$

$$\frac{l^2/mk}{r} = 1 + \frac{l^2 A}{mk}\cos(\theta - \theta')$$

here $l^2/mk = J$ & $\frac{l^2 A}{mk} = e$

$$J/r = 1 + e\cos(\theta - \theta')$$

Differentiating eqn ⑤, we get

$$-\frac{1}{r^2}\dot{r} = -A\sin(\theta-\theta')\dot{\theta}$$

$$-\frac{\dot{r}}{r^2} = -A\sin(\theta-\theta')\frac{l}{mr^2} \qquad \because \dot{\theta} = \frac{l}{mr^2}$$

$$\dot{r} = \frac{Al}{m}\sin(\theta-\theta') \rightarrow ⑥$$

From the first integrals $\dot{\theta} = \frac{l}{mr^2}$

$$\frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r} = E \rightarrow ⑦$$

Sub. eqn ⑥ in ⑦, we have the following eqn

$$\frac{1}{2}\cancel{m}\frac{A^2l^2}{m^2}\sin^2(\theta-\theta') + \frac{l^2}{2m}\left[\frac{mk}{l^2} + A\cos(\theta-\theta')\right]^2 \quad \frac{l^2}{2m}\left(\frac{m^2k^2}{l^4}\right)$$

$$-k\left(\frac{mk}{l^2} + A\cos(\theta-\theta')\right) = E \rightarrow ⑧$$

$$\frac{A^2l^2}{2m} + \frac{mk^2}{2l^2} - \frac{mk^2}{l^2} = E$$

$$\frac{A^2l^2}{2m} - \frac{1}{2}\frac{mk^2}{l^2} = E$$

$$\frac{A^2l^2}{2m} = \frac{2E\cancel{l^2}}{\cancel{m}k^2}\ \frac{mk^2}{2l^2} + E$$

$$A^2 = \frac{2m}{l^2}\left(\frac{mk^2}{2l^2} + E\right)$$

$$= \frac{m^2k^2}{l^4} + \frac{2mE}{l^2} \quad \times \div mk^2l^2$$

$$A^2 = \frac{m^2k^2}{l^4} + \frac{2m^2k^2El^2}{mk^2l^4}$$

$$A^2 = \left(\frac{mk}{l^2}\right)^2\left(1 + \frac{2El^2}{mk^2}\right)$$

$$A = \frac{mk}{l^2}\sqrt{1 + \frac{2El^2}{mk^2}}$$

(or)

$$\frac{Al^2}{mk} = \sqrt{1 + \frac{2El^2}{mk^2}}$$

$$e = \sqrt{1 + \frac{2El^2}{mk^2}} \rightarrow ⑨$$

Thus, when a particle is moving under inverse square law of force, its orbit is represented by eqn ⑤ $\left(\frac{1}{r} = \frac{mk}{l^2} + A\cos(\theta-\theta') \text{ (or) } 1 + e\cos(\theta-\theta') \right)$

This is the general eqn of conic section with one focus at the origin, and the eccentricity e is given by eqn ④. $\left[\because \frac{Al^2}{mk} = e\right.$ $e = \sqrt{1 + \frac{2El^2}{mk^2}}$

The magnitude of e decides the nature of orbit.

Value of e	Value of E	Conic
$e > 1$	$E > 0$	Hyperbola
$e = 1$	$E = 0$	Parabola
$e < 1$	$E < 0$	ellipse
$e = 0$	$E = -\frac{mk^2}{2l^2}$	circle

For a circular orbit, T and V are const in time, and from the Virial theorem $E = T + V = -\frac{V}{2} + V = \frac{V}{2}$

Hence

$$E = -\frac{k}{2r} \rightarrow ⑩ \qquad \because V = -\frac{k}{r}$$

From ~~energy~~ the eqn $f(r) = -\frac{l^2}{mr^3}$, the stmt of equilibrium b/w the central force and the effective force, we can write

$$\frac{k}{r^2} = \frac{l^2}{mr^3}$$

$$\because f(r) = -l^2/mr^3 \rightarrow -\frac{\partial V}{\partial r} = -\frac{l^2}{mr^3}$$

$$-\frac{\partial}{\partial r}\left(\frac{k}{r}\right) = -l^2/mr^3$$

$$-\frac{k}{r^2} = -l^2/mr^3$$

(or)

$$r = l^2/mk \rightarrow ⑪$$

Sub ⑪ in ⑩,

$$E = -\frac{mk^2}{2l^2} \rightarrow ⑫$$

This is the condition for circular motion.

In case of elliptic orbits, it can be shown the major axis depends solely upon the energy. The semimajor axis is one-half the sum of the two apsidal distances r_1 & r_2.

For $e<1$ or $E<0$, the orbit is elliptical, gn. by

$$\frac{J}{r} = 1 + e\cos(\theta - \theta') \rightarrow (13)$$

here $J = l^2/mk$ & $e = \sqrt{1 + \frac{2El^2}{mk^2}}$

when $\theta - \theta' = 0$ (or) $\cos(\theta - \theta') = 1$ $r = r_1$

$\theta - \theta' = \pi$ (or) $\cos(\theta - \theta') = -1$ $r = r_2$

Sub. these conditions in eqn (13)

$$\frac{J}{r_1} = 1 + e \quad \text{(or)} \quad r_1 = \frac{J}{1+e} \rightarrow (14)$$

$$\frac{J}{r_2} = 1 - e \quad \text{(or)} \quad r_2 = \frac{J}{1-e} \rightarrow (15)$$

The Semi-major axis of an ellipse is gn. by

$$a = \frac{r_1 + r_2}{2} = \frac{J}{2}\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = \frac{J}{2}\left(\frac{1+\cancel{e}+1-\cancel{e}}{1-e^2}\right)$$

$$= \frac{\cancel{2}J}{\cancel{2}(1-e^2)}$$

$$=$$

$$a = \frac{J}{1-e^2} \rightarrow (16)$$

$$a = -\frac{\cancel{l^2}}{\cancel{mk}} \cdot \frac{m\cancel{k^2}}{2E\cancel{l^2}}$$

$$a = -\frac{k}{2E} \rightarrow (17)$$

$$E = -\frac{k}{2a} \rightarrow (18)$$

$\because J = l^2/mk$ $e = \sqrt{1 + \frac{2El^2}{mk^2}}$

$e^2 = \left(1 + \frac{2El^2}{mk^2}\right)$

$1 - e^2 = -\frac{2El^2}{mk^2}$

Sub. eqn (18) in eqn (9)

$$e = \sqrt{1 - \frac{l^2}{mka}}$$

$$e^2 = 1 - \frac{l^2}{mka}$$

$$\frac{l^2}{mka} = 1 - e^2$$

$\because e = \sqrt{1 + \frac{2El^2}{mk^2}}$

$= 1 + \frac{\cancel{2}\cdot(-\cancel{k}/\cancel{2}a)\, l^2}{mk^2}$

$e = \sqrt{1 - \frac{l^2}{mka}}$

$$\frac{l^2}{mk} = a(1-e^2) \rightarrow (19)$$

From (13),

$$\frac{J}{r} = 1+e \cos(\theta-\theta')$$

$J = a(1-e^2)$ $\because$ From eqn (16)

$$a(1-e^2) = r(1+e\cos(\theta-\theta'))$$

The eqn for the elliptical orbit can be written as

$$\boxed{r = \frac{a(1-e^2)}{1+e\cos(\theta-\theta')}}$$